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Optimal exercise of American options under stock pinning

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Abstract

We address the problem of optimally exercising American options based on the assumption that the underlying stock's price follows a Brownian bridge whose final value coincides with the strike price. In order to do so, we solve the discounted optimal stopping problem endowed with the gain function $G(x) = (S - x)^+$ and a Brownian bridge whose final value equals S . These settings came up as a first approach of optimally exercising an option within the so-called “stock pinning” scenario. The optimal stopping boundary for this problem is proved to be the unique solution, up to certain regularity conditions, of an integral equation, which is then numerically solved by an algorithm hereby exposed. We face the case where the volatility is unspecified by providing an estimated optimal stopping boundary that, alongside with pointwise confidence intervals, provide alternative stopping rules. Finally, we demonstrate the usefulness of our method within the stock pinning scenario through a comparison with the optimal exercise time based on a geometric Brownian motion. We base our comparison on the contingent claims and the 5-minutes intraday stock price data of Apple and IBM for the period 2011–2018. Supplementary materials with the main proofs and auxiliary lemmas are available online.

Keywords: American option; Brownian bridge; free-boundary problem; optimal stopping; option pricing; put-call parity; stock pinning.

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1 Introduction

An option is a contingent claim in which the holder has the right to either sell (*put option*) or buy (*call option*) the underlying asset for a previously agreed upon value, called the *exercise price* or the *strike price*. The *American option* allows the holder to exercise at any point up to a *maturity date*, also called *exercise date*, whereas the *European option* only admits exercising at maturity. This increased flexibility makes American contingent claims an appealing security for investors and increases the complexity of their pricing.

The first attempt on pricing American contingent claims was that of McKean (1965), where he proved that the discounted *Optimal Stopping Problem* (OSP) relative to an American call option can be set as a free-boundary problem, whose solution was proved to satisfy a countable system of integral equations, although he did not tackle the existence and uniqueness of a solution for such a system. A major drawback of this approach is to rely on the derivative of the *Optimal Stopping Boundary* (OSB) by using it explicitly in the integral equations that arose from the free-boundary problem, which makes the system intractable. Despite McKean (1965) recognized the problem of finding the fair price of an American option as an OSP, it was not until Bensoussan (1984) and Karatzas (1988) that this relation was backed up with financial arguments. A few years later Kim (1990), Jacka (1991), and Carr et al. (1992) independently proved that the fair value of an American option can be split into the sum of the price of the corresponding European option and the so-called *early exercise premium* (i.e., the price an option's holder has to pay for the right to exercise the option before the maturity date), thereby enhancing the economic interpretation of the pricing of an American contingent claim. From that decomposition, a tractable non-linear integral equation having the OSB as a solution was derived, hence it was called the *free-boundary equation*, but whether or not this solution is unique remained an open problem until Peskir (2005b) demonstrated that the free-boundary equation by itself actually characterizes the OSB, relying mainly in an extension of the Itô's formula appeared in Peskir (2005a).

When pricing American options, it is commonly assumed that stock price dynamics follow a geometric Brownian motion law. This belief might be compromised when it comes

to the so-called *stock pinning* or *pinning-at-the-strike* effect, i.e., the phenomenon describing the tendency of the price of *optionable* stocks (stocks with available options) to end up near the strike price of some of its underlying options at the expiration date. Researchers concerning the pinning-at-the-strike scenario embrace the idea that not only the stock's price determines the option's value, as it is addressed in the classical option pricing theory, but the other way around is also valid, especially right before expiration days. Therefore the works regarding this phenomenon are mainly of two types: searching and exposing clear evidence that supports the pinning-at-the-strike behavior; and developing models that successfully fits the stock's price dynamics under a pinning-at-the-strike scenario.

The first pinning model backed up with financial arguments came with Avellaneda and Lipkin (2003), who developed a *Stochastic Differential Equation* (SDE) for stock price dynamics based on the assumption that the pinning behavior was mainly driven by delta hedging long option positions. This assumption, along three other possible explanations, was explored by Ni et al. (2005), who additionally provided sound evidence supporting that the price of optionable stocks tends to go near the strike price on expiration dates, while this behavior does not take place neither among non-optionable stocks nor optionable stocks on non-expiration dates. Further readings on the pinning-at-the-strike scenario and more recent developments of pinning models can be find in Jeannin et al. (2008) and Avellaneda et al. (2012).

In this work we offer a pricing formula for American put options under the assumption that the underlying stock's price follows a Brownian bridge process whose final value is the strike price S . We do it by solving the associated discounted OSP by considering the gain function $G(x) = (S - x)^+$ and the aforementioned underlying process. These settings can be seen as a first approach to pricing an American put option under the stock pinning effect. Following a method similar to that of Peskir (2005b), we get the corresponding free-boundary equation and prove that the OSB is its unique solution, up to some regularity conditions. Moreover, we include the non-discounted case in our analysis by allowing the discount rate to be exactly zero, therefore extending the methodology of Peskir (2005b) for positive discount rates. We then show how to easily extend these results for American

call options. Likewise, we prove how our OSB is modified, in the non-discounted scenario, when the gain function is changed for the identity. This last case was already addressed by Shepp (1969) and Ekström and Wanhörp (2009), who provided a closed form for the OSB that allows us to test an algorithm hereby exposed for computing our OSB, as well as an inferential method to provide confidence curves for it when the volatility of the underlying process is estimated via maximum likelihood. Finally, we study the new OSB in a real dataset based on options of Apple's and IBM's equities, evidencing that our proposal is competitive with Peskir (2005b)'s OSB based on a geometric Brownian motion, especially when the stock's price exhibits a pinning-at-the-strike behavior.

The rest of the paper is structured as follows. Section 2 settles the problem and presents general comments on OSPs. Section 3 is split in two parts: Subsection 3.1 exposes the required results for obtaining the free-boundary equation and shows that the OSB is its unique solution; whereas Subsection 3.2 shows how the solution of an OSP changes when the gain function and the underlying process are slightly modified. Section 4 deals with the problem of computing the OSP and the uncertainty associated to the estimation of the process' volatility. In Section 5 the profits generated when optimally exercising Apple's and IBM's options according to our OSB and the OSB based on the geometric Brownian motion. The proofs of the theoretical results and auxiliary lemmas are relegated to the supplementary materials.

2 Problem setting

Suppose that an American put option have been bought at time $t > 0$ with strike price $S > 0$ and maturity date $T > t$. Denote by $X^{[t,T]} := (X_{t+s})_{s=0}^{T-t}$ the stochastic process representing the stock's dynamics from t to T , and whose probability law is denoted as $\mathbb{P}_{t,x}$ to emphasize that $X_t = x$, where x is a point in the state space of $X^{[t,T]}$. Then, the (discounted) potential payoff of the American put option at time $t + s$, $0 \leq s \leq T - t$, is given by $e^{-\lambda s} G(X_{t+s})$, where $G(x) = (S - x)^+$ is called the gain function and $\lambda \geq 0$ denotes the risk-free interest rate currently held by the market. One can find the arbitrage-free price

of the American option (Bensoussan, 1984; Karatzas, 1988) by solving the discounted OSP

$$V(t, x) = \sup_{0 \leq \tau \leq T-t} \mathbb{E}_{t,x} [e^{-\lambda \tau} G(X_{t+\tau})], \quad (1)$$

where V is called the value function, $\mathbb{E}_{t,x}$ is the expectation with respect to $\mathbb{P}_{t,x}$, and the supreme above is taken over all the stopping times of $X^{[t,T]}$.

We assume that the process $X^{[t,T]}$ is a Brownian bridge with unknown volatility, and final value equal to S . A Brownian bridge approach offers a relative simple framework while reasonably fits a pinning-at-the-strike scenario. Hence, we will assume throughout this paper that $X^{[t,T]}$ satisfies the following SDE,

$$dX_{t+s} = \frac{S - X_{t+s}}{T - t - s} ds + \sigma dW_s, \quad 0 \leq s \leq T - t, \quad (2)$$

which yields the solution

$$X_{t+s} = X_t \frac{T - t - s}{T - t} + S \frac{s}{T - t} + \sigma \sqrt{\frac{T - t - s}{T - t}} W_s, \quad (3)$$

where $(W_s)_{s \geq 0}$ denotes a standard Brownian motion. Recall that a Brownian bridge can be seen as a Brownian motion conditioned to its final value. In this sense, it turns out that the former process inherits the volatility of the last one, while the drift information is lost after conditioning, i.e., $[W_t^{(1)} \mid W_T^{(1)} = S] \stackrel{d}{=} [W_t^{(2)} \mid W_T^{(2)} = S]$, where $(W_t^{(i)})$ is such that $dW_t^{(i)} = \mu_i dt + \sigma dW_t$, where $\mu_i \in \mathbb{R}$, $\sigma > 0$, $i = 1, 2$. The infinitesimal generator of the Brownian bridge $X = X^{[0,T]}$ is the operator \mathbb{L}_X , which takes a suitably smooth function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and returns the function

$$(\mathbb{L}_X f)(t, x) = \frac{S - x}{T - t} \partial_x f(t, x) + \frac{\sigma^2}{2} \partial_{x^2} f(t, x), \quad (4)$$

A solution of an OSP like (1) must be regarded as both the value $V(t, x)$ and a so-called *Optimal Stopping Time* (OST) $\tau^*(t, x)$, satisfying

$$V(t, x) = \mathbb{E}_{t,x} [e^{-\lambda \tau^*(t,x)} G(X_{t+\tau^*(t,x)})]. \quad (5)$$

Under quite mild conditions, namely V being lower semi-continuous and G upper semi-continuous (see Corollary 2.9 from Peskir and Shiryaev (2006)), it is guaranteed that an

OSP of the form (1) has the OST $\tau^*(t, x)$ given by the first hitting time of $X^{[t, T]}$ into the so-called *stopping set* D under $\mathbb{P}_{t, x}$, i.e.,

$$\tau^*(t, x) := \inf\{0 \leq s \leq T - t : X_{t+s} \in D \mid X_t = x\}, \quad (6)$$

where D is the closed set

$$D := \{(t, x) \in [0, T] \times \mathbb{R} : V(t, x) = G(x)\}. \quad (7)$$

The complement of the stopping set is called the *continuation set*, having the form

$$C := \{(t, x) \in [0, T] \times \mathbb{R} : V(t, x) > G(x)\}. \quad (8)$$

A common approach for solving an OSP like (1) is to reformulate it as the following free-boundary problem, for the unknowns V and ∂C (optimal stopping boundary),

$$\partial_t V + \mathbb{L}_X V = \lambda V \quad \text{on } C, \quad (9)$$

$$V > G \quad \text{on } C, \quad (10)$$

$$V = G \quad \text{on } D, \quad (11)$$

$$V_x = G_x \quad \text{on } \partial C, \quad (12)$$

where (9), (10), and (11) easily come after the definition of D , C , and $\tau^*(t, x)$ (see Proposition 2), whereas (12) (*smooth fit condition*) depends on how good-behaved the OSB is. For deeper insights on the relation between OSPs and free-boundary problems, we refer the interested reader to Peskir and Shiryaev (2006).

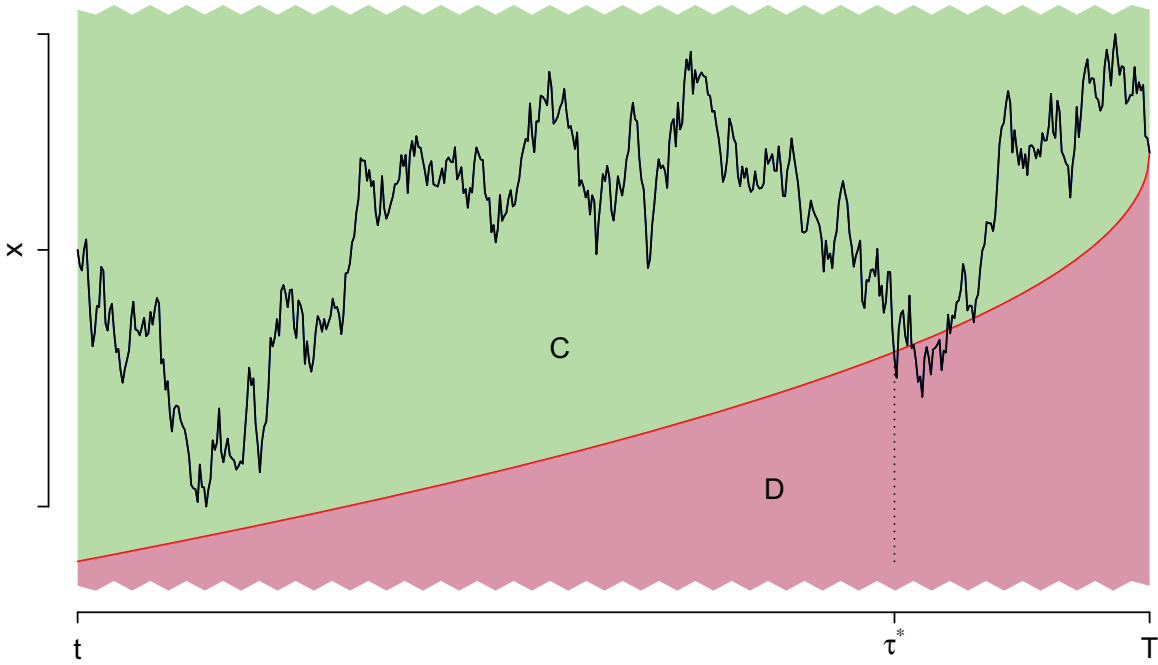


Figure 1: Main components of an OSP: stopping set (D), continuation set (C), optimal stopping boundary (red curve), and optimal stopping time ($\tau^*(t, x)$).

The next section is devoted to solve (1) by solving the associated free-boundary problem (9)–(12).

3 Theoretical results

3.1 Pricing American put options for a Brownian bridge

Since $G(x) = (S - x)^+$ is continuous, we can guarantee, by applying Corollary 2.9 in conjunction with Remark 2.10 from Peskir and Shiryaev (2006), that the OSP (1) has the OST $\tau^*(t, x)$ defined at (6). The next proposition sheds light on the form of D and C on the space $[0, T] \times \mathbb{R}$. It also proves some regularity properties about the OSB that we will use later to derive the smooth fit condition.

Proposition 1 *There exists a non-decreasing right-continuous function $b : [0, T] \rightarrow \mathbb{R}$ such that $b(t) < S$ for all $t \in [0, T)$, $b(T) = S$, and $D = \{(t, x) \in [0, T] \times \mathbb{R} : x \leq b(t)\}$.*

We obtain the free-boundary problem associated to the OSP (1) in the following proposition. Conditions (10) and (11) are not addressed as they come straightforwardly from (8) and (7), respectively. Furthermore, we prove certain smoothing conditions about V for a twofold purpose: to get the continuity of b in Proposition 3, and to be able to use the extension of the Itô's formula exposed in the supplementary document in order to derive the pricing formula for V (18) and ultimately the free-boundary equation (20).

Proposition 2 *The value function V from (1) satisfies:*

(i) V is $\mathcal{C}^{1,2}$ on C and on D , and $\partial_t V + \mathbb{L}_X V = \lambda V$ on C .

(ii) $x \mapsto V(t, x)$ is convex and strictly increasing for all $t \in [0, T]$. Moreover,

$$\partial_x V(t, x) = \mathbb{E} \left[e^{-\lambda \tau^*(t, x)} \frac{T - t - \tau^*(t, x)}{T - t} \right]. \quad (13)$$

(iii) The smooth fit condition holds, i.e., $\partial_x V(t, b(t)) = -1$ for all $t \in [0, T]$.

(iv) $t \mapsto V(t, x)$ is non-increasing for all $x \in \mathbb{R}$.

(v) V is continuous.

The next proposition tackles the continuity of the OSB. It is worth to say that, due to the explicit form for V_x showed at (13), we can obtain the continuity of b for the special case $\lambda = 0$. This situation is the only one with known closed-form solution (see Subsection 3.2), therefore providing a benchmark for the numerical experiments of Section 4.

Proposition 3 *The optimal stopping boundary b for the OSP (1) is continuous.*

Throughout Propositions 1, 2, and 3 we have gathered the required conditions in order to apply the Itô's formula extension exposed in the supplement to the function $F(s, x) = e^{-\lambda s} V(t + s, x)$, from where we get that

$$\begin{aligned} e^{-\lambda s} V(t + s, X_{t+s}) &= V(t, X_t) + \int_0^s e^{-\lambda u} (\partial_t V + \mathbb{L}_X V - \lambda V)(t + u, X_{t+u}) du \\ &\quad + \int_0^s \sigma e^{-\lambda u} \partial_x V(t + u, X_{t+u}) dB_u. \end{aligned} \quad (14)$$

Notice that the above formula is missing the local time term due to the continuity of $x \mapsto \partial_x V(t, x)$ for all $t \in [0, T]$.

Recalling that $\partial_t V + \mathbb{L}_X V = \lambda V$ on C and $(\partial_t V + \mathbb{L}_X V - \lambda V)(t, x) = -(S - x) \left(\frac{1}{T-t} + \lambda \right)$ for all $(t, x) \in D$, taking $\mathbb{P}_{t,x}$ -expectation (causing the vanishing of the martingale term), setting $s = T - t$, and making a simple change of variable in the integral, we get from (14) the following pricing formula for the American put option:

$$V(t, x) = \int_t^T e^{-\lambda(u-t)} \left(\frac{1}{T-u} + \lambda \right) \mathbb{E}_{t,x} [(S - X_u) \mathbb{1}(X_u \leq b(u))] du. \quad (15)$$

We know from (3) that, for $u \in [t, T]$, $X_u^{[t,T]} \sim \mathcal{N}(\mu(t, x, u), \nu_\sigma^2(t, u))$ under $\mathbb{P}_{t,x}$, where

$$\mu(t, x, u) = x \frac{T-u}{T-t} + S \frac{u-t}{T-t}, \quad (16)$$

$$\nu_\sigma(t, u) = \sigma \sqrt{\frac{(u-t)(T-u)}{T-t}}. \quad (17)$$

For any random variable X we have that $\mathbb{E}[X \mathbb{1}(X \leq a)] = \mathbb{P}[X \leq a] \mathbb{E}[X | X \leq a]$. In addition, if $X \sim \mathcal{N}(\mu, \nu^2)$, then $\mathbb{E}[X | X \leq a] = \mu - \frac{\nu \phi(z)}{\Phi(z)}$, where $z = \frac{a-\mu}{\nu}$, and ϕ and Φ denote, respectively, the density function and the distribution function of a standard normal random variable. Putting this together and after some algebraic rearrangements, we obtain a more tractable representation for V than the one exposed at (15):

$$V(t, x) = \int_t^T K_{\sigma,\lambda}(t, x, u, b(u)) du, \quad (18)$$

where

$$\begin{aligned} K_{\sigma,\lambda}(t, x_1, u, x_2) &= e^{-\lambda(u-t)} \left(\frac{1}{T-u} + \lambda \right) \\ &\times [(S - \mu(t, x_1, u)) \Phi(z_\sigma(t, x_1, u, x_2)) + \nu_\sigma(t, u) \phi(z_\sigma(t, x_1, u, x_2))], \end{aligned} \quad (19)$$

with

$$z_\sigma(t, x_1, u, x_2) = \frac{x_2 - \mu(t, x_1, u)}{\nu_\sigma(t, u)}.$$

Since $V(t, x) = S - x$ for all $(t, x) \in D$, we can take $x \uparrow b(t)$ on both sides in the equation (18) in order to obtain the following type two Volterra non-linear integral equation for the OSB b :

$$b(t) = S - \int_t^T K_{\sigma,\lambda}(t, b(t), u, b(u)) du. \quad (20)$$

It turns out that the OSB not only satisfies the Volterra integral equation (20), but additionally is its only solution up to some regularity conditions.

Theorem 1 *The optimal stopping boundary for problem (1) is characterized as the unique solution of the type two non-linear Volterra integral equation (20), within the class of continuous functions of bounded variation $c : [0, T] \rightarrow \mathbb{R}$ such that $c(t) < S$ for all $t \in (0, T)$.*

3.2 Put-call parity and other extensions

All the results stated in the Subsection 3.1 have their own analog when it comes to pricing the American call option, this is, to solve the OSP (1) but this time endowed with the gain function $G(x) = (x - S)^+$. Despite obtaining the optimal exercise strategy for both the call and put contingent claims should be of the same degree of complexity and follows similar arguments, it turns out that there is a direct relation between both stopping sets (and therefore the optimal stopping boundaries). The next proposition not only sheds light upon that relation, but also establishes the connection between the stopping sets of two general OSPs up to some regularity conditions.

Proposition 4 *Let $G_i, V_i, X^{(i)} = (X_{t+s}^{(i)})_{s=0}^{T-t}$, and D_i be, respectively, the gain function, the value function, the underlying process, and the stopping set associated to the OSP*

$$V_i(t, x) = \sup_{0 \leq \tau \leq T-t} \mathbb{E}_{t,x} \left[e^{-\lambda \tau} G_i \left(X_{t+\tau}^{(i)} \right) \right], \quad (21)$$

where $(t, x) \in [0, T] \times \mathbb{R}$, $\lambda \in \mathbb{R}$, $i = 1, 2$. Suppose that V_i is lower semi-continuous and G_i is upper semi-continuous. Then, the following relations hold:

- (i) If $G_1(x) = G_2(\alpha x + A)$ and $\left[\alpha X_{t+s}^{(1)} + A \mid X_t^{(1)} = x \right] \stackrel{d}{=} \left[X_{t+s}^{(2)} \mid X_t^{(2)} = \alpha x + A \right]$ for some constants $A \in \mathbb{R}$, $\alpha \neq 0$, and for all $s \in [0, T-t]$, $x \in \mathbb{R}$, then $D_1 = \{(t, x) \in [0, T] \times \mathbb{R} : (t, \alpha x + A) \in D_2\}$.
- (ii) If $G_1 = G_2$ on $D_1 \cup D_2$ and $\left[X_{t+s}^{(1)} \mid X_t^{(1)} = x \right] \stackrel{d}{=} \left[X_{t+s}^{(2)} \mid X_t^{(2)} = x \right]$ for all $s \in [0, T-t]$ and $x \in \mathbb{R}$, then $D_1 = D_2$.

The following corollary exposes some extensions of the work done in Subsection 3.1 while illustrating with examples how to use the tools provided by Proposition 4.

Corollary 1 *Consider the two OSPs*

$$V_i(t, x) = \sup_{0 \leq \tau \leq T-t} \mathbb{E}_{t,x} \left[e^{-\lambda \tau} G_i \left(X_{t+\tau}^{(i)} \right) \right],$$

where both $X^{(i)} = (X_{t+s}^{(i)})_{s=0}^{T-t}$ are Brownian bridges with $X_T^{(i)} = S_i$, $i = 1, 2$. Denote by $b_i : [0, T] \rightarrow \mathbb{R}$ to the OSB for the i -th OSP. Then:

(i) If $\lambda \geq 0$, $G_1(x) = (x - S_1)^+$, $G_2(x) = (S_2 - x)^+$, and $S_1 = S_2$, then $b_1(t) = 2S_i - b_2(t)$ for $t \in [0, T]$ and $i = 1, 2$.

(ii) If $\lambda = 0$, $G_1(x) = x$, $G_2(x) = (S_2 - x)^+$, and $S_1 = 0$, then $b_2(t) = S_2 - b_1(t)$ for $t \in [0, T]$.

Statement (i) of Corollary 1 reveals the put-call parity under the Brownian bridge assumption. Thus, the OSB for the American put option is just a reflection, with respect to the strike price axis, of the OSB for the American call option.

On the other hand, statement (ii) from Corollary 1 relates the OSB b defined at (20) for $\lambda = 0$ and the OSB b_I coming from a non-discounted OSP with the identity gain function and a Brownian bridge ending up at zero as the underlying process. This OSP was solved for the first time by Shepp (1969) and later studied by Ekström and Wanhörp (2009). Both provided the explicit solution for b_I when the volatility of the Brownian bridge is $\sigma = 1$, which trivially generalizes to $b_I(t) = \sigma B \sqrt{T - t}$ for $\sigma > 0$, where $B \approx 0.8399$. Shepp (1969) did so by using a Brownian motion representation of the Brownian bridge, while Ekström and Wanhörp (2009), based on Shepp (1969)'s result, transformed the associated free-boundary problem into an ordinary differential equation by assuming a parametrization of the OSB and a particular form for the value function.

4 Boundary computation and inference

4.1 Solving the free-boundary equation

The lack of an explicit solution for the free-boundary equation (20) demands a numerical approach for computing the OSB. Let $(t_i)_{i=0}^N$ be a partition of the interval $[0, T]$ for some

$N \in \mathbb{N}$. The method we are about to show is based on a proposal made by Pedersen and Peskir (2002), which suggested to approximate the integral in the free-boundary equation (20) by a right Riemann sum, hence enabling the computation of the value of $b(t_i)$, for $i = 0, \dots, N-1$, by using only the values $b(t_j)$, with $j = i+1, \dots, N$. Therefore, by knowing the value of the boundary at the last point ($b(t_N) = b(T) = S$), we can obtain its value at the second last point $b(t_{N-1})$ and recursively construct the whole OSB evaluated at $(t_i)_{i=0}^N$.

Under our settings, the right Riemann sum is no longer a valid option because we know from (19) that, depending on the shape of the boundary b near the expiration date, $K(t_i, b(t_i), u, b(u))$ could explode as $u \rightarrow T$, so we cannot evaluate the kernel K at the right point in the last subinterval $(t_{N-1}, T]$. To deal with this issue, we employ a right Riemann sum approximation along all the subintervals except the last one, ending up with the following discrete version of the Volterra integral equation (20):

$$b(t_i) \approx S - \sum_{j=i+1}^{N-1} (t_j - t_j) K_{\sigma, \lambda}(t_i, b(t_i), t_j, b(t_j)) - I(t_i, t_{N-1}), \quad (22)$$

for $i = 0, 1, \dots, N-1$, where $I(t_i, t_{N-1}) := \int_{t_{N-1}}^T K_{\sigma}(t_i, b(t_i), u, b(u)) du$. However, it can be shown that $0 \leq I(t_i, t_{N-1}) \leq H(t_i, t_{N-1})$, where

$$\begin{aligned} H(t_i, t_{N-1}) &:= e^{-\lambda(t_{N-1}-t_i)} \int_{t_{N-1}}^T (1 + \lambda(T-u)) \left(\frac{S-b(t_i)}{T-t_i} + \sigma \sqrt{\frac{1}{2\pi(T-u)}} \right) du \\ &= e^{-\lambda(t_{N-1}-t_i)} \left((S-b(t_i)) \frac{T-t_{N-1}}{T-t_i} \left(1 + \frac{\lambda}{2}(T-t_{N-1}) \right) + \right. \\ &\quad \left. \sigma \sqrt{\frac{2(T-t_{N-1})}{\pi}} \left(1 + \frac{\lambda}{3}(T-t_{N-1}) \right) \right), \end{aligned}$$

by using (16) and (17), the form of the kernel (19), and the fact that $\Phi(x) \leq 1$ and $\phi(x) \leq (2\pi)^{-1/2}$ for all $x \in \mathbb{R}$.

Therefore, $I(t_i, t_{N-1}) \approx H(t_i, t_{N-1})/2$ is a reasonable approximation, as we can provide an upper bound for the error $\varepsilon(t_i, t_{N-1}) := |H(t_i, t_{N-1})/2 - I(t_i, t_{N-1})|$, namely $\varepsilon(t_i, t_{N-1}) \leq H(t_i, t_{N-1})/2$. Moreover, $\varepsilon(t_i, t_{N-1}) = \mathcal{O}(\sqrt{T-t_{N-1}})$ as $t_{N-1} \rightarrow T$, that is, there exist a positive constant M_i such that $\varepsilon(t_i, t_{N-1}) \leq M_i \sqrt{T-t_{N-1}}$ for all t_{N-1} sufficiently close to T . Hence, we get the following after substituting $I(t_i, t_{N-1})$ for $H(t_i, t_{N-1})/2$ in (22) and

making some rearrangements:

$$b(t_{N-1}) \approx \left(\frac{1}{2} - \frac{\lambda}{4}(T - t_{N-1}) \right)^{-1} \times \left(\frac{S}{2} \left(1 - \frac{\lambda}{2}(T - t_{N-1}) \right) - \sigma \sqrt{\frac{T - t_{N-1}}{2\pi}} \left(1 + \frac{\lambda}{3}(T - t_{N-1}) \right) \right), \quad (23)$$

$$b(t_i) \approx \left(1 - \frac{1}{2}e^{-\lambda(t_{N-1}-t_i)} \left(1 + \frac{\lambda}{2}(T - t_{N-1}) \right) \frac{T - t_{N-1}}{T - t_i} \right)^{-1} \times \left(S - \sum_{j=i+1}^{N-1} (t_j - t_j) K_{\sigma,\lambda}(t_i, b(t_i), t_j, b(t_j)) - \frac{1}{2}e^{-\lambda(t_{N-1}-t_i)} \left(S \frac{T - t_{N-1}}{T - t_i} \left(1 + \frac{\lambda}{2}(T - t_{N-1}) \right) + \sigma \sqrt{\frac{2(T - t_{N-1})}{\pi}} \left(1 + \frac{\lambda}{3}(T - t_{N-1}) \right) \right) \right). \quad (24)$$

The procedure for computing the estimated boundary according to the previous approximations is laid down in Algorithm 1. From now on we will use \tilde{b} to denote the cubic-spline interpolating curve that goes through the numerical approximation of the boundary, via Algorithm 1, at the given points $(t_i)_{i=0}^N$.

Algorithm 1: Optimal stopping boundary computation

Input: $S, \lambda, (t_i)_{i=0}^N, \delta$

Output: $(\tilde{b}(t_i))_{i=0}^N$

Code:

$\tilde{b}(T) \leftarrow S$

Update $\tilde{b}(t_{N-1})$ according to (23)

for $i = N - 2$ **to** 0 **do**

$\tilde{b}(t_i) \leftarrow \tilde{b}(t_{i+1})$

$\varepsilon \leftarrow 1$

while $\varepsilon > \delta$ **do**

$\tilde{b}_{\text{old}}(t_i) \leftarrow \tilde{b}(t_i)$

 Update $\tilde{b}(t_i)$ according to (24)

$\varepsilon \leftarrow |\tilde{b}_{\text{old}}(t_i) - \tilde{b}(t_i)| / |\tilde{b}_{\text{old}}(t_i)|$

end

end

Recall that Corollary 1 proved that the OSB b_I completely determines the OSB b_0 , this is, the function b defined at (20) with $\lambda = 0$. Specifically, $b_0(t) = S - b_I(t) = S - B\sigma\sqrt{T - t}$. Having the explicit form of b_0 allowed us to test the accuracy of Algorithm 1 and to tune up its parameters in order to fit b_0 better. For example, we empirically determined that $\delta = 10^{-3}$ offered a good trade-off between accuracy and computational expediency. This value was considered every time Algorithm 1 was employed.

Furthermore, we empirically addressed the issue of how to choose the partition in order to get a good approximation of the boundary while not compromising many computational resources. Two conclusions arose from our investigations: first, for a uniform partition, errors increase near the expiration date T , which points out to the need of a higher density of nodes over there; second, given $N + 1$ nodes, a partition whose distances between consecutive nodes smoothly narrow as t_i gets closer to T works better than one whose distances narrow following non-smooth patterns. Therefore, we decided to use the logarithmically-spaced grid $t_i = \log\left(1 + \frac{i}{N}(e^T - 1)\right)$, $i = 0, \dots, N$, with $N = 200$, every time we computed

\tilde{b} . Figure 2 shows how precise Algorithm 1 is by comparing the computed boundary \tilde{b}_0 versus its explicit form for $S = 10$, $T = 1$, $\lambda = 0$, and $\sigma = 1$.

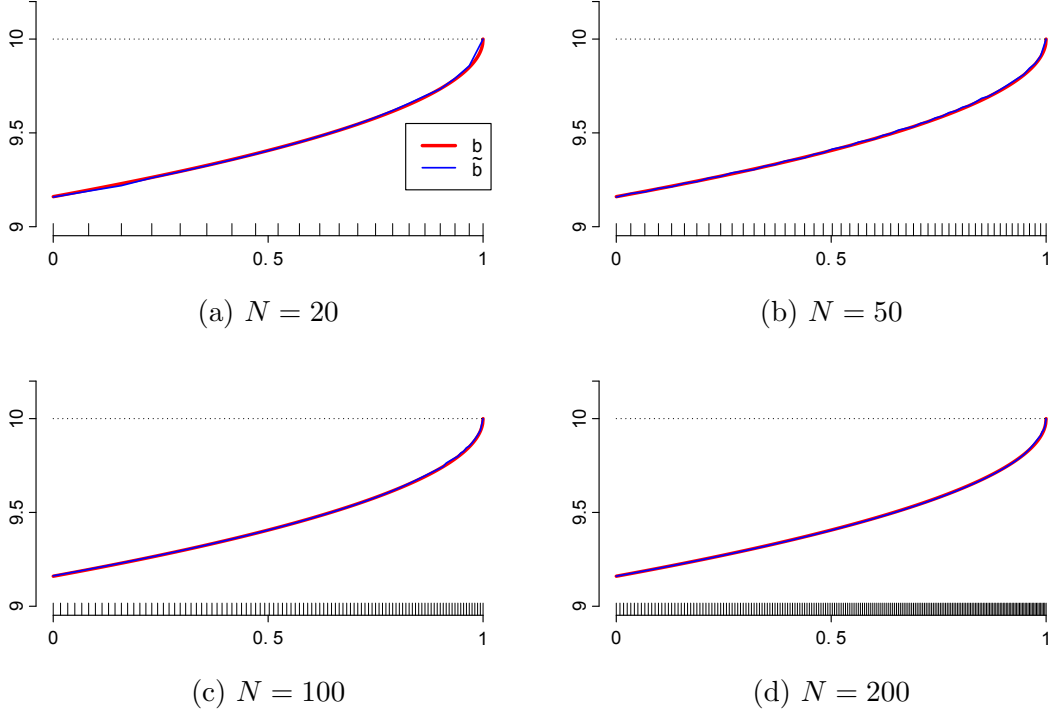


Figure 2: Boundary estimation via Algorithm 1 for different partition sizes and for the parameters $S = 10$, $T = 1$, $\lambda = 0$, and $\sigma = 1$. The estimation becomes more accurate as the partition gets thinner.

4.2 Estimating the volatility

In real life scenarios the volatility of the assumed underlying process is unknown. Therefore, in order to provide the OSB, we first need to learn the volatility based on past data.

Assume we are able to record the process values along the points $t_0 = 0 < t_1 < \dots < t_{N-1} < t_N = T$, for $N \in \mathbb{N}$, so at time t_n , with $n \in \{0, 1, \dots, N\}$, we have gathered a sample $(X_{t_i})_{i=0}^n$ from the historical path of the Brownian bridge $(X_t)_{t=0}^T$ such that $X_T = S$. From (3), we have that

$$X_{t_i} \mid X_{t_{i-1}} \sim \mathcal{N}(\mu(t_{i-1}, X_{t_{i-1}}, t_i), \nu_\sigma^2(t_{i-1}, t_i)), \quad i = 1, \dots, n,$$

where μ and ν_σ are given by (16) and (17), respectively. Therefore, given the Markovian structure of the process, the log-likelihood function of the volatility has the form

$$\ell(\sigma \mid (t_i, X_{t_i})_{i=0}^n) = C - n \log(\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^n \left(\frac{X_{t_i} - \mu(t_{i-1}, X_{t_{i-1}}, t_i)}{\nu_1(t_{i-1}, t_i)} \right)^2,$$

where C is a constant independent from σ . Differentiating with respect to σ yields the Maximum Likelihood Estimator (MLE) for σ :

$$\hat{\sigma}_n = \sqrt{\frac{1}{n} \sum_{i=1}^n \left(\frac{X_{t_i} - \mu(t_{i-1}, X_{t_{i-1}}, t_i)}{\nu_1(t_{i-1}, t_i)} \right)^2}.$$

If we consider the equally spaced partition ($t_i = i\frac{T}{N}$, $i = 0, 1, \dots, N$), we can guarantee from Dacunha-Castelle and Florens-Zmirou (1986) the following convergence in law:

$$\sqrt{n}(\hat{\sigma}_n - \sigma) \rightsquigarrow \mathcal{N}(0, I(\sigma)^{-1}),$$

when $n \rightarrow \infty$ (hence $N \rightarrow \infty$) and $T \rightarrow \infty$ such that $t_i - t_{i-1} = T/N$ remains constant, with $i = 1, \dots, N$. The Fisher information readily follows as $I(\sigma) = \frac{2}{\sigma^2}$.

4.3 Confidence intervals for the boundary

In this subsection we find bounds, with a given degree of confidence, for the uncertainty propagated from the estimation of σ towards the numerical approximation of the OSB via Algorithm 1. In order to do so, we assume that the OSB is differentiable, so we are allowed to apply the delta method, under the same asymptotic conditions described in the previous subsection, to ensure that

$$\sqrt{n}(b_{\hat{\sigma}_n}(t) - b_\sigma(t)) \rightsquigarrow \mathcal{N}\left(0, \left(\frac{\partial b_\sigma}{\partial \sigma}(\sigma, t)\right)^2 I(\sigma)^{-1}\right), \quad (25)$$

where b_σ represents the OSB defined at (20) associated to a process with volatility σ . Hence, we can plug-in the estimation $\hat{\sigma}_n$ into (25) to come up with the following asymptotic $100(1 - \alpha)\%$ (pointwise) *confidence curves* for b_σ :

$$(c_{1, \hat{\sigma}_n}(t), c_{2, \hat{\sigma}_n}(t)) := \left(b_{\hat{\sigma}_n}(t) \pm \frac{z_{1-\alpha/2}}{\sqrt{nI(\hat{\sigma}_n)}} \left| \frac{\partial b_\sigma}{\partial \sigma}(t) \right|_{\sigma=\hat{\sigma}_n} \right), \quad (26)$$

where $z_{1-\alpha/2}$ represents the $\alpha/2$ -upper quantile of a standard normal distribution. Algorithm 1 can be used to compute an approximation of the term $\frac{\partial b_\sigma}{\partial \sigma}(\cdot)$ by means of the difference quotient $(b_{\hat{\sigma}_n+\varepsilon}(\cdot) - b_{\hat{\sigma}_n}(\cdot))/\varepsilon$ for some small $\varepsilon > 0$. We denote by $(\tilde{c}_{1,\hat{\sigma}_n}(t), \tilde{c}_{2,\hat{\sigma}_n}(t))$ the approximation of the confidence interval (26) coming from this approach at $t \in [0, T]$. Through the paper we use $\varepsilon = 10^{-2}$, which was empirically designed to provide, along with $\delta = 10^{-3}$ for Algorithm 1, a good compromise between accuracy, stability, and computational expediency on the approximation of the confidence curves. Figure 3 illustrates, for one path of a Brownian bridge, how the boundary estimation and its confidence curves work, while Figure 4 empirically validates the confidence curves approximations by computing the proportion, out of $M = 1000$ trials, of non-inclusions of the true boundary within the confidence curves.

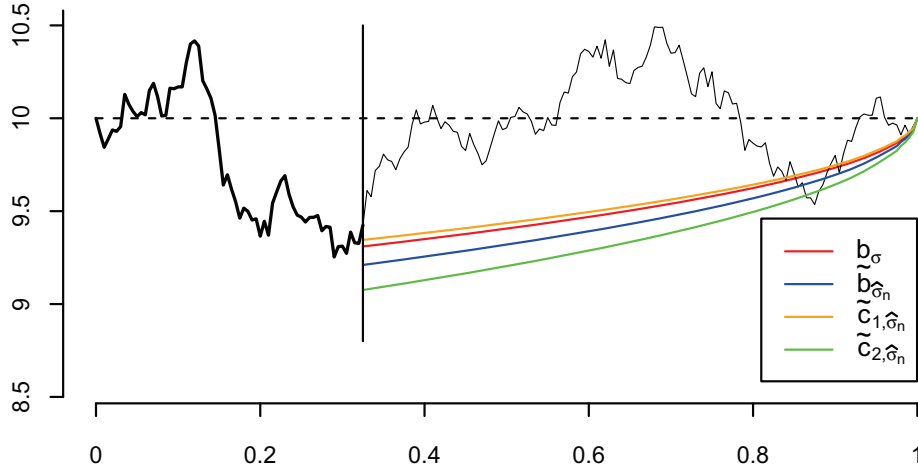


Figure 3: Inferring the boundary using one third ($n = 66$, $N = 200$) of the Brownian bridge path, for $T = 1$, $S = 10$, $X_0 = 10$, $\lambda = 0$, and $\sigma = 1$. The solid curves represent the true boundary b_σ (red curve), estimated boundary $\tilde{b}_{\hat{\sigma}_n}$ (blue curve), upper confidence curve $\tilde{c}_{1,\hat{\sigma}_n}$ (orange curve), and lower confidence curve $\tilde{c}_{2,\hat{\sigma}_n}$ (green curve).

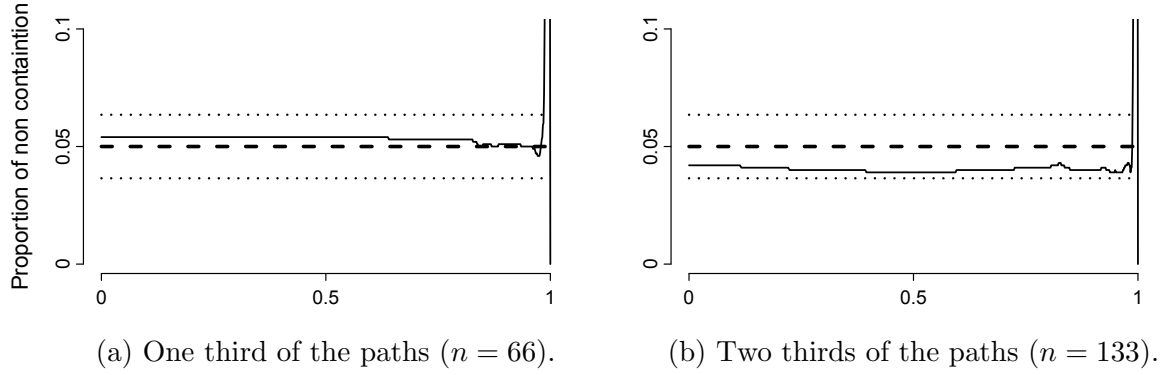


Figure 4: Pointwise proportion of non-inclusions of the true boundary within the confidence curves, using $S = 10$, $X_0 = 10$, $T = 1$, $\lambda = 0$, $\sigma = 1$, and $\alpha = 0.05$. The proportion is an estimation of the confidence level α , based on $M = 1000$ Brownian bridges trajectories. For each path, one third (a) or two thirds (b) of the observations were used to compute σ and then estimate the confidence curves (26). The continuous line represents the proportion of non inclusion, the dashed line stands for the confidence level α and the dotted lines are placed at the values $\alpha \pm z_{0.975} \sqrt{\frac{\alpha(1-\alpha)}{M}}$. The spikes at $T = 1$ are numerical artifacts related with the zero-variance of $\tilde{b}_{\hat{\sigma}_n}(T)$.

Notice on Figure 4 how the proportion spikes near the last point $t_N = T = 1$, meaning that the true boundary rarely lies within the confidence curves at those points. This numerical artifact happens because the confidence curves have zero-variance at the maturity date T (actually $\tilde{c}_{1,\hat{\sigma}_n}(T) = \tilde{c}_{2,\hat{\sigma}_n}(T) = S$), and the numerical approximation of $b(t_{N-1})$ given at (23) is slightly biased, thus affecting the accuracy of the estimated boundary and leaving the true boundary outside the confidence curves most of the times near maturity. However, notice that this drawback is negligible in practise, since the estimated boundary and the confidence curves are very close to the true boundary in terms of *absolute* distance.

4.4 Simulations

Once we can perform inference for the true OSB, some immediate questions arise: how much optimality is lost by $\tilde{b}_{\hat{\sigma}_n}$ when compared with b_σ ? How do the stopping strategies associated to the curves $\tilde{c}_{i,\hat{\sigma}_n}$, $i = 1, 2$, compare with the one for $\tilde{b}_{\hat{\sigma}_n}$? For example, a risk-averse (or risk-lover) strategy would be to consider the upper (lower) confidence curve

$\tilde{c}_{1,\hat{\sigma}_n}$ ($\tilde{c}_{2,\hat{\sigma}_n}$) as the stopping rule, being this the most conservative (liberal) option within the uncertainty on estimating b_σ , and a balanced strategy would consider the estimated boundary $\tilde{b}_{\hat{\sigma}_n}$. In this section we investigate empirically how these stopping strategies behave, assuming $\sigma = 1$, $T = 1$, $S = 10$, $X_0 = 10$, and $\lambda = 0$. To do so, we first estimate the payoff associated to each of them, and then we compare these payoffs with the one generated by considering the true boundary $b_0(t) = S - B\sigma\sqrt{T-t}$, this is, the OSB defined at (20) for $\lambda = 0$.

Note that the choice of $\sigma = 1$ is not restrictive. Indeed, let $X = (X_s)_{s=0}^T$ be a Brownian bridge going from $X_0 = x$ to $X_T = S$ with σ volatility. Since for a standard Brownian motion $(W_s)_{s \geq 0}$ the time scaling $(a^{1/2}W_{s/a})$ results into a standard Brownian motion for $a > 0$, then (3) entails that $Y = (\sigma^{-1/2}X_{s/\sigma})_{s=0}^{\sigma T}$ is a Brownian bridge from $Y_0 = \sigma^{-1/2}x$ to $Y_T = \sigma^{-1/2}S$ with volatility 1. If we call b_X and b_Y to the OSBs associated to X and Y respectively, the relation $b_Y(t) = \sigma^{-1/2}b_X(t/\sigma)$ holds. Therefore, the OSB for Y is just a time-scaling of the one for X and the simulation results for $\sigma \neq 1$ follow by a time-scaling of the ones for $\sigma = 1$.

To perform the comparison we first defined a subset of $[0, T] \times \mathbb{R}$ where the payoffs were computed. Clearly, given $X_0 = X_T = 10$, some pairs $(t, x) \in [0, T] \times \mathbb{R}$ are very unlikely to be visited by the Brownian bridge $(X_s)_{s=0}^T$. Therefore, we decided to carry out the comparison along the pairs $(t_i, X_{t_i}^{(q)})$, for $i = 1, \dots, N$, $N = 200$, and $q = 0.2, 0.4, 0.6, 0.8$, where $t_i = i\frac{T}{N}$ and $X_{t_i}^{(q)}$ represents the q -quantile of a $\mathcal{N}(0, \frac{t_i(T-t_i)}{T})$, the marginal distribution of the process at time t_i . Notice from Figure 5 how the quantile curves capture the behavior of the variability of the process.

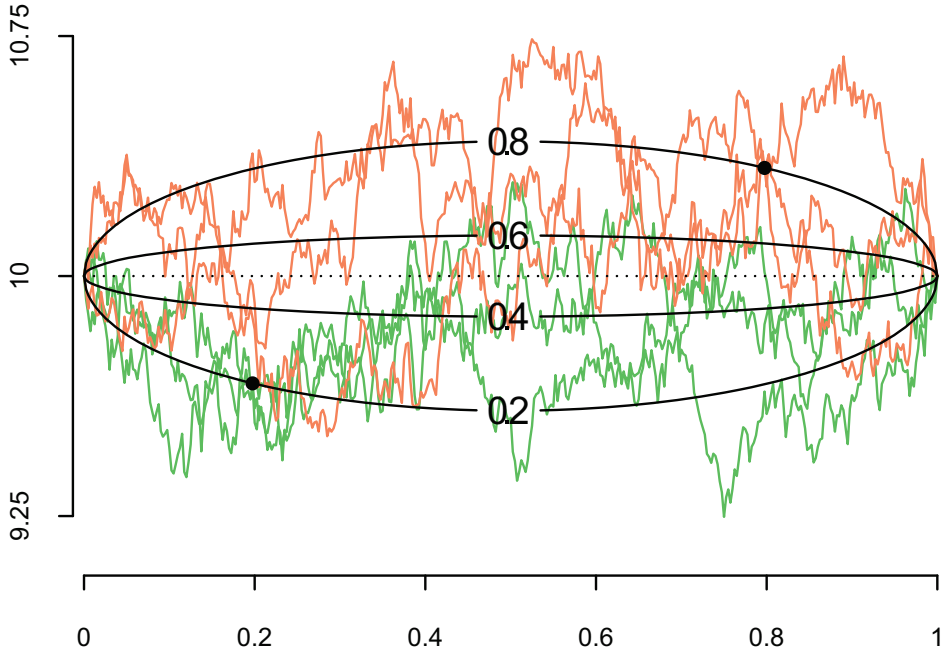


Figure 5: $X_t^{(q)}$ for $q = 0.2, 0.4, 0.6, 0.8$, where $X_t^{(q)}$ is the q -quantile of a $\mathcal{N}(0, t(1-t))$, the marginal distribution, at time t , of a Brownian bridge with unit volatility such that $X_0 = X_1 = 10$. The green lines are paths from $(X_t | X_{0.2} = X_{0.2}^{(0.2)})_{t=0}^T$, while the orange lines are trajectories from $(X_t | X_{0.8} = X_{0.8}^{(0.8)})_{t=0}^T$, with $X_{0.2}^{0.2} \approx 9.6649$ and $X_{0.8}^{0.8} \approx 10.3382$.

For each i and q , we generated $M = 1000$ different paths $(s_j, X_{s_j})_{j=0}^{rN}$ of a Brownian bridge with volatility $\sigma = 1$ and going from $(0, 0)$ to $(1, 0)$. Each path was sampled at times $s_j = j \frac{T}{rn}$, for $j = 0, 1, \dots, rN$, for $r = 1$ and $r = 25$. The idea behind this setting is to tackle both the low-frequency scenario, which regards investors with access to daily prices or even smaller amount of data, and the high-frequency scenario, addressing high volumes of information as it happens to be when recording intraday prices. We forced each path to pass through $(t_i, X_{t_i}^{(q)})$ (see Figure 5), and used the past $(s_j, X_{s_j})_{j=0}^{r_i}$ of each trajectory to estimate the boundary and the confidence curves. The future $(s_j, X_{s_j})_{j=r_i}^{rN}$ was employed to gather M observations of the payoff associated to each stopping rule, whose means and variances are shown in Figures 6 and 7 respectively.

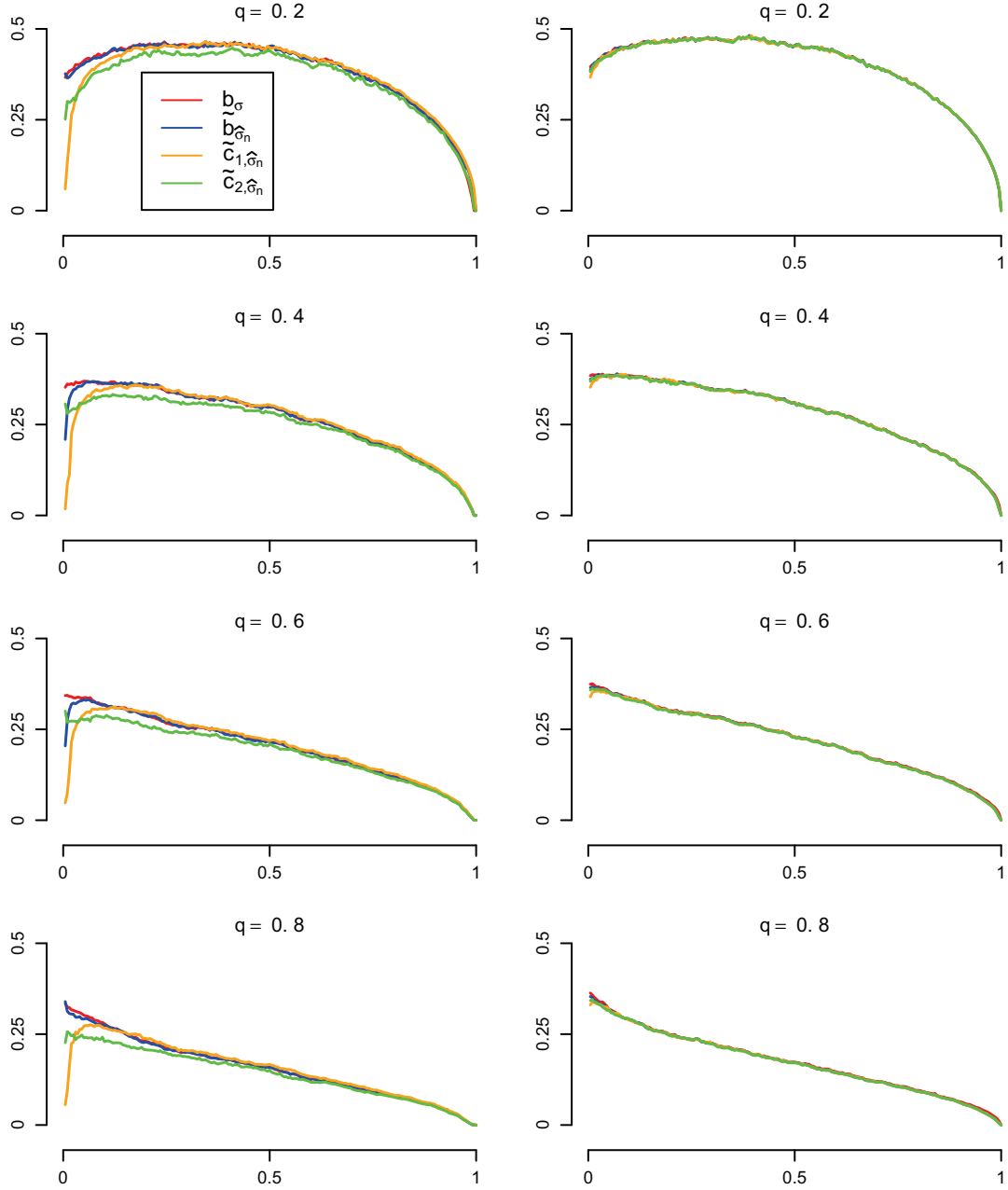


Figure 6: Mean of the payoff associated to: true boundary b_σ (red curve), estimated boundary $\tilde{b}_{\hat{\sigma}_n}$ (blue curve), upper confidence curve $\tilde{c}_{1,\hat{\sigma}_n}$ (orange curve), lower confidence curve $\tilde{c}_{2,\hat{\sigma}_n}$ (green curve). The left column shows the low-frequency ($r = 1$) scenario, while the right one stands for the high-frequency ($r = 25$) scenario. We used $\sigma = 1$, $T = 1$, $S = 10$, $X_0 = 10$, and $\lambda = 0$.

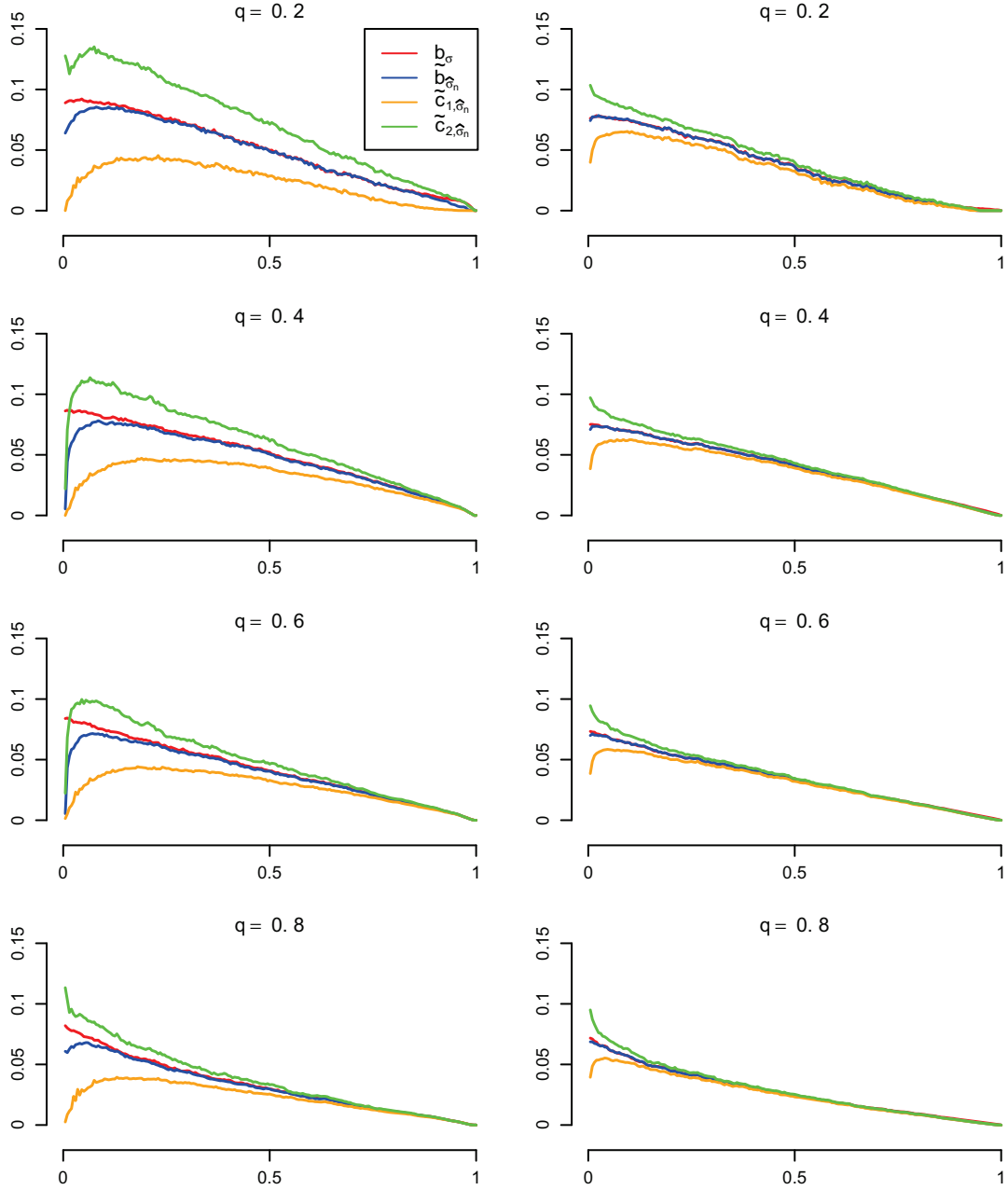


Figure 7: Variance of the payoff associated to: true boundary b_σ (red curve), estimated boundary $\tilde{b}_{\hat{\sigma}_n}$ (blue curve), upper confidence curve $\tilde{c}_{1,\hat{\sigma}_n}$ (orange curve), lower confidence curve $\tilde{c}_{2,\hat{\sigma}_n}$ (green curve). The left column shows the low-frequency ($r = 1$) scenario, while the right one stands for the high-frequency ($r = 25$) scenario. We used $\sigma = 1$, $T = 1$, $S = 10$, $X_0 = 10$, and $\lambda = 0$.

Figure 6 shows the value functions associated to each stopping rule. The red curve

being the one associated to the OSB. An important fact revealed by Figure 6 is that, both in the low-frequency and high-frequency scenarios, with just a few initial observations of the asset's trajectory, for which the estimate $\hat{\sigma}_n$ becomes reliable, the estimate $\tilde{b}_{\hat{\sigma}_n}$ behaves almost indistinguishable to b_σ in terms of the mean payoff.

Despite the variance payoff is not the optimal criterion taken into account in the OSP (1), it is worth to know how it behaves for the three different stopping strategies, as it represents the risk associated to adopt each stopping rule as the OSB. Clearly, for any pair (t, x) , the higher the stopping boundary one assumes, the smaller payoff variance is obtained. Figure 7 not only reflects this behavior by suggesting the upper confidence curve as the best stopping strategy, but also reveals that the variance does exhibit considerable differences for the stopping rules in the low-frequency scenario, and these differences increase both when the time get closer to the initial point $t = 0$, and the quantile level q decreases. However, in the high-frequency scenario this effect is alleviated, since when n increases the confidence intervals narrow and get close to $\tilde{b}_{\hat{\sigma}_n}$ (and this one to b_σ) and thus the three stopping strategies yield similar results. Figures 6 and 7 also reveal that both the mean and variance payoff associated to the estimated boundary $\tilde{b}_{\hat{\sigma}}$ converge to the ones associated to the true boundary as more data is taken for estimating σ .

The pragmatic bottom line of the simulation study can be condensed in the following rules-of-thumb: if $15 < n < 1000$, it is advised to adopt the upper confidence curve as the stopping rule, because $\tilde{c}_{1, \hat{\sigma}_n}$ has almost the same mean payoff of all the other stopping rules while having considerable less variance payoff; if $n \geq 1000$ the mean and variance payoff of all the three stopping rules are quite similar, being the most efficient option to just assume $\tilde{b}_{\hat{\sigma}_n}$ without computing the confidence curves, therefore saving computational resources.

For $n \leq 15$ the best candidate for the OSB is not obvious, and it would depend on which criterion one decides to take into account for measuring the mean-variance trade-off of all the three strategies.

5 Pinning at the strike and real data study

We compare in this section the performance of the optimal stopping strategy under the Brownian bridge assumption with the classical geometric Brownian motion approach (Peskir,

2005b), which does not take into account the insight information of the asset's price at maturity. We do so in a real data study for various scenarios with different extents of pinning-at-the-strike effects.

Since it has been noticed that the pinning behavior is more likely to take place among heavily traded options, we built our data around Apple's and IBM's equities. We considered all the options expiring within the span of January 2011 – September 2018, a total of 8905 options for Apple and 4833 for IBM. We generically denote by M the total number of options of each company, and for the j -th option we denote by $(X_{t_i}^{(j)})_{i=0}^{N_j}$ the 5-minutes tick close price of the underlying stock during its lifespan after dividing by its strike price S_j . In order to quantify the strength of the pinning effect for each one of those path prices, we define the *pinning deviance* as $p_j := |X_{t_{N_j}}^{(j)} - 1|$, with $j = 1, \dots, M$. Under perfect pinning, $X_{t_{N_j}}^{(j)} = 1$ and $p_j = 0$.

Our OSB works under the pinning assumption, but in reality, knowing beforehand if a stock will be pinned is not trivial. Despite the forecasting of the pinning is not the scope of this paper (for a systematic treatment, the reader is referred to Avellaneda and Lipkin (2003), Jeannin et al. (2008), and Avellaneda et al. (2012)), we provide some evidence pointing towards the possibility of actually forecasting pinning by means of the trading volume of the options associated to a stock. For that, we studied the association between the pinning deviances $(p_j)_{j=1}^M$ and the number of open contracts for a given option or *Open Interest* (OI). We do so through the *weighted* OI for options expiring within 2017, $\text{wOI}_j := \sum_{k=0}^{K_j} w_{j,k} o_{j,k}$, where $o_{j,k}$ is the OI of the j -th option at day k after it was opened, K_j is the total number of days the option remained available, and the weights $w_{j,k} := e^{-(1-k/K_j)} / \sum_{i=0}^{K_j} e^{-(1-i/K_j)}$, $j = 1, \dots, M$, give more importance to OIs closer to the maturity date. Recall that the wOI is an *observable* quantity that can be used to forecast the pinning deviance. The Spearman's rank correlation coefficient scored -0.5932 for Apple's options and -0.4281 for IBM's options. Figure 8 shows a non-parametric regression estimating the relationship among the pinning deviance and the weighted OI by using a local linear kernel estimator with adaptive bandwidth based on k -nearest neighbours,

where k was selected by least squares cross validation (see Hayfield and Racine (2008)). The decreasing trend of the regression curve corroborates the negative correlation between the weighted OI and the pinning index.

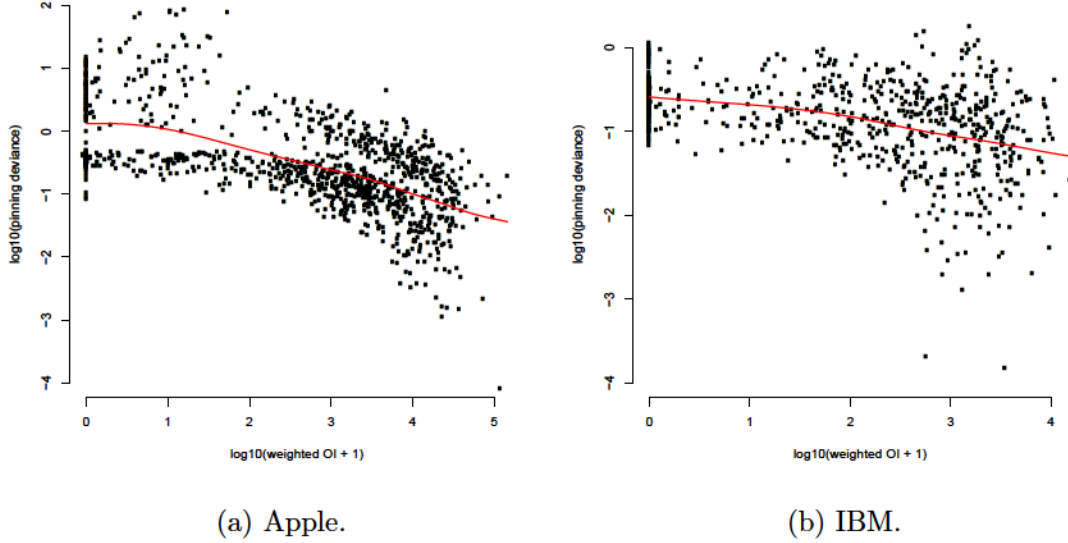


Figure 8: Non-parametric regression relating the variables “pinning deviance” and “weighted OI” for options expiring within 2017. The fit was done on the transformed data $(\log_{10}(p_i))_{i=1}^M$ and $(\log_{10}(1 + \text{wOI}_j))_{j=1}^M$.

The setting for the performed application is described next. We split each path $(X_{t_i}^{(j)})_{i=0}^{N_j}$ into two subsets: the first $\rho 100\%$ of the prices $(X_{t_i}^{(j)})_{i=0}^{\lfloor \rho N_j \rfloor}$, called the historical set, and the last remaining prices (including the present time) $(X_{t_i}^{(j)})_{i=\lfloor \rho N_j \rfloor}^{N_j}$, named the future set, for $\rho \in \mathcal{P} = \{0.1, 0.2, \dots, 0.9\}$ and $j = 1, \dots, M$. Therefore $1 - \rho$ represents the proportion of life time for a given option. We used the historical set to estimate the volatility of both models. Then we fixed the risk-free interest rate $\lambda_{j,\rho}$ as the 52 weeks treasury bill rate (extracted from U.S. Department of the Treasury (2018)) held by the market when the split of $(X_{t_i}^{(j)})_{i=0}^{N_j}$ was done. Unlike the Brownian bridge process, the geometric Brownian motion model accounts for a drift parameter. This drift was set to the risk-free interest rate in order to make the discounted process a martingale.

Afterwards, we computed the estimated OSBs according to both approaches. The Brownian bridge boundary defined at (20) was computed via Algorithm 1, while the geo-

metric Brownian motion boundary tackled at Peskir (2005b) was computed following the method exposed in Pedersen and Peskir (2002, page 12). Both numerical approaches are quite similar, the only subtle difference relying on the fact that, for the Brownian bridge, the last part of the integral is computed as outlined in Algorithm 1, while the geometric Brownian motion requires no special treatment. The settings considered for running Algorithm 1 were $S = 1$ (the stock's prices were standardized by the strike prices), $T = 1$ (all the maturity dates were standardized to 1), 201 nodes for the time partition, and the ones described in Subsection 4.1. We took into account the confidence curves exposed in Subsection 4.3 for the comparison, but as we are working in a high-frequency sampling scenario (the average sample size is 8402), both confidence curves provide almost the same results as the estimated boundary, so in order to avoid redundancy we did not include these results.

Finally, the future set was used in order to get the profit generated by optimally exercising the option within the remaining time, this is, we computed $e^{-\lambda_{j,\rho}\tau_{\text{BB}}^{j,\rho}}(1 - X_{t_{\lfloor \rho N_j \rfloor} + \tau_{\text{BB}}^{j,\rho}}^{(j)})$ and $e^{-\lambda_{j,\rho}\tau_{\text{GBM}}^{j,\rho}}(1 - X_{t_{\lfloor \rho N_j \rfloor} + \tau_{\text{GBM}}^{j,\rho}}^{(j)})$, where $\tau_{\text{BB}}^{j,\rho}$ and $\tau_{\text{GBM}}^{j,\rho}$ are the OSTs associated, respectively, to the Brownian bridge and geometric Brownian motion strategies under the initial condition $(t_{\lfloor \rho N_j \rfloor}, X_{t_{\lfloor \rho N_j \rfloor}}^{(j)})$. We defined then the following “ ρ -aggregated” cumulative profit for measuring the goodness of both methods:

$$\begin{aligned}\text{BB}(p) &= \frac{1}{|\mathcal{P}||\mathcal{J}(p)|} \sum_{j \in \mathcal{J}(p)} \sum_{\rho \in \mathcal{P}} e^{-\lambda_{j,\rho}\tau_{\text{BB}}^{j,\rho}} (1 - X_{t_{\lfloor \rho N_j \rfloor} + \tau_{\text{BB}}^{j,\rho}}^{(j)}), \\ \text{GBM}(p) &= \frac{1}{|\mathcal{P}||\mathcal{J}(p)|} \sum_{j \in \mathcal{J}(p)} \sum_{\rho \in \mathcal{P}} e^{-\lambda_{j,\rho}\tau_{\text{GBM}}^{j,\rho}} (1 - X_{t_{\lfloor \rho N_j \rfloor} + \tau_{\text{GBM}}^{j,\rho}}^{(j)}),\end{aligned}$$

where $\mathcal{J}(p) := \{j = 1, \dots, M : p_j < p\}$, and $|\mathcal{P}|$ and $|\mathcal{J}(p)|$ are the number of elements in \mathcal{P} and $\mathcal{J}(p)$, respectively. Figure 9 displays the results of the relative profit $(\text{BB}(p) - \text{GBM}(p)) / \text{GBM}(p)$.

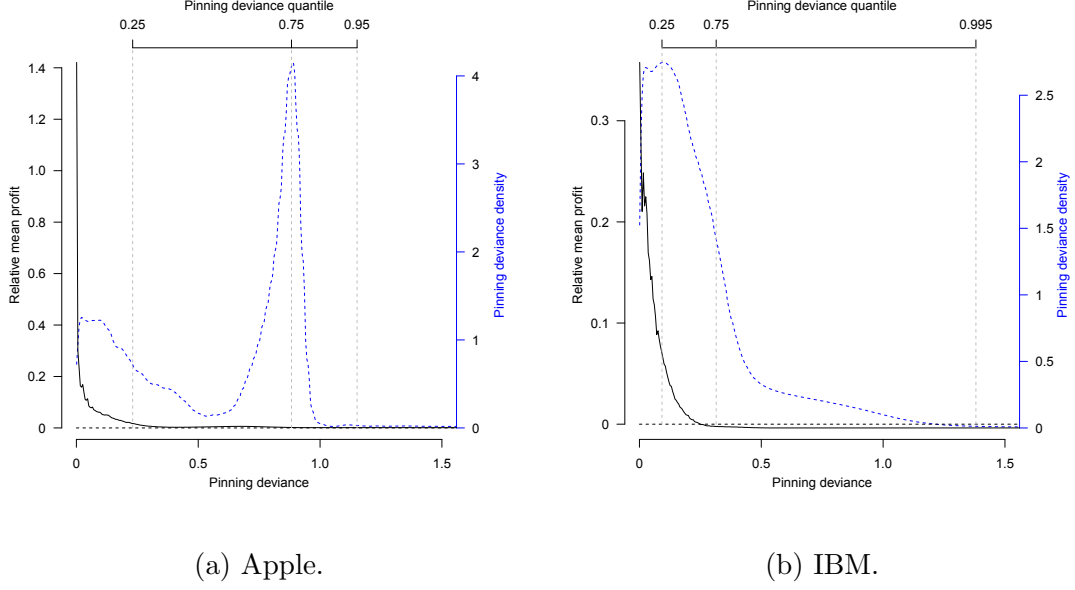


Figure 9: Results of the real data application. For a pinning deviance p in the x -axis the black curve is the relative profit $(BB(p) - GBM(p)) / GBM(p)$, while the blue dashed curve represents the kernel density estimation of the pinning deviances evaluated at p .

Figure 9 suggests that our Brownian bridge proposal behaves better for options with low pinning deviance than the geometric Brownian motion approach. As expected, this advantage fades away as we take distance from an ideal pinning-at-the-strike scenario, this is, the pinning deviance goes higher. However, note that we outperform the geometric Brownian motion method for Apple’s options even when the whole dataset is used, while for IBM our advantage remains for up to about the 60% of the options with lower pinning deviances.

Since the focus of this paper is to find the optimal strategy to exercise an option according to (1), we do not tackle the problem of *when* is advantageous to buy an option, so the options’ price were not included to compute the profits in Figure 9. However, if we were to take them into account, the sign of the relative mean curve showed in Figure 9 would remain the same as long as both approaches buy the options at the same time, which means that our Brownian bridge approach would still outperform the geometric Brownian motion scenario for the same cases that already did, although the strength of this advantage might change considerably depending on the strategy for buying options.

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Supplement to “Optimal exercise of American options under stock pinning”

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Abstract

This supplement is structured as follows. Section A contains all the proofs omitted in the paper. Section B states technical lemmas required by these proofs.

Keywords: American option; Brownian bridge; free-boundary problem; optimal stopping; option pricing; put-call parity; stock pinning.

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A Main proofs

Proof of Proposition 1. Take an admissible pair (t, x) satisfying $x \geq S$ and $t < T$, and consider the stopping time $\tau_\varepsilon := \inf \{0 \leq s \leq T - t : X_{t+s} \leq S - \varepsilon \mid X_t = x\}$ (assume for convenience that $\inf \{\emptyset\} = T - t$), for $\varepsilon > 0$. Notice that $\mathbb{P}_{t,x}[\tau_\varepsilon < T - t] > 0$, which implies that $V(t, x) \geq \mathbb{E}_{t,x}[e^{-\lambda\tau_\varepsilon}G(X_{t+\tau_\varepsilon})] > 0 = G(x)$, from where it comes that $(t, x) \in C$.

Define $b(t) := \sup \{x \in \mathbb{R} : (t, x) \in D\}$. The above arguments guarantee that $b(t) < S$ for all $t \in [0, T)$, and we get from (1) that $b(T) = S$. Furthermore, from (1) it can be easily noticed that as λ increases $V(t, x)$ decreases and therefore $b(t)$ increases, and since $b(t)$ is known to be finite for all t when $\lambda = 0$ (see Subsection 3.2), then we can guarantee that $b(t) > -\infty$ for all values of λ .

Notice that, since D is a closed set, $b(t) \in D$ for all $t \in [0, T]$. In order to prove that D has the form claimed in Proposition 1, let us take $x < b(t)$ and consider the OST $\tau^* = \tau^*(t, x)$. Then, relying on (1), (3) and (5), we get

$$\begin{aligned} V(t, x) - V(t, b(t)) &\leq \mathbb{E}_{t,x}[e^{-\lambda\tau^*}G(X_{t+\tau^*})] - \mathbb{E}_{t,b(t)}[e^{-\lambda\tau^*}G(X_{t+\tau^*})] \\ &\leq \mathbb{E}_{t,0} \left[\left(X_{t+\tau^*} + b(t) \frac{T-t-\tau^*}{T-t} - X_{t+\tau^*} - x \frac{T-t-\tau^*}{T-t} \right)^+ \right] \\ &= (b(t) - x) \mathbb{E} \left[\frac{T-t-\tau^*}{T-t} \right] \\ &\leq b(t) - x, \end{aligned} \tag{27}$$

where we used the relation

$$G(a) - G(b) \leq (b - a)^+, \tag{28}$$

for all $a, b \in \mathbb{R}$, for the second inequality. Since $V(t, b(t)) = S - b(t)$, we get from the above relation that $V(t, x) \leq S - x = G(x)$, which means that $(t, x) \in D$ and therefore $\{(t, x) \in [0, T] \times \mathbb{R} : x \leq b(t)\} \subset D$. On the other hand, if $(t, x) \in D$, then $x \geq b(t)$, which proves the reverse inclusion.

Take now $t, t' \in [0, T]$ and $x \in \mathbb{R}$ such that $t' < t$ and $(t, x) \in C$, then, since the function $t \mapsto V(t, x)$ is non-increasing for all $x \in \mathbb{R}$ (see (iv) from Proposition 2), $V(t', x) \geq V(t, x) > G(x)$, i.e., $(t', x) \in C$. Hence, b is non-decreasing.

Finally, in order to prove the right-continuity of b , let us fix $t \in (0, T)$ and notice that, since b is non-decreasing, then $b(t^+) \geq b(t)$. On the other hand, as D is a closed set and $(t+h, b(t+h)) \in D$ for all $0 < h \leq T-t$, then $(t^+, b(t^+)) \in D$ or, equivalently, $b(t^+) \leq b(t)$. ■

Proof of Proposition 2. (i) Half of the statement relies on the result obtained in Peskir and Shiryaev (2006, Section 7.1) relative to the Dirichlet problem. Specifically, it states that V is $\mathcal{C}^{1,2}$ on C and $\partial_t V + \mathbb{L}_X V = \lambda V$ on C . On the other hand, since $V(t, x) = G(x) = S - x$ for all $(t, x) \in D$, V is $\mathcal{C}^{1,2}$ also on D .

(ii) We easily get the convexity of $x \mapsto V(t, x)$ by plugging-in (3) into (1). To prove (13) let us fix an arbitrary point $(t, x) \in [0, T] \times \mathbb{R}$, consider $\tau^* = \tau^*(t, x)$ and $\tau_\varepsilon = \tau^*(t, x + \varepsilon)$ for some $\varepsilon \in \mathbb{R}$. Since $\tau_\varepsilon \rightarrow \tau$ a.s., by arguing similarly to (27), we get

$$\begin{aligned} \varepsilon^{-1}(V(t, x + \varepsilon) - V(t, x)) &\leq -\mathbb{E} \left[e^{-\lambda \tau_\varepsilon} \frac{T - t - \tau_\varepsilon}{T - t} \right] \\ &\xrightarrow{\varepsilon \rightarrow 0} -\mathbb{E} \left[e^{-\lambda \tau^*} \frac{T - t - \tau^*}{T - t} \right], \end{aligned} \quad (29)$$

where the limit is valid due to the dominated convergence theorem. For $\varepsilon < 0$ the reverse inequality emerges, giving us, after making $\varepsilon \rightarrow 0$, the relation $\partial_x^- V(t, x) \leq -\mathbb{E} \left[e^{-\lambda \tau^*} \frac{T - t - \tau^*}{T - t} \right] \leq \partial_x^+ V(t, x)$, which, due to the continuity of $x \mapsto \partial_x V(t, x)$ on $(-\infty, b(t))$ and on $(b(t), \infty)$ for all $t \in [0, T]$ (V is $\mathcal{C}^{1,2}$ on C and on D), turns into $\partial_x V(t, x) = -\mathbb{E} \left[e^{-\lambda \tau^*} \frac{T - t - \tau^*}{T - t} \right]$ for all (t, x) where $t \in [0, T]$ and $x \neq b(t)$. For $x = b(t)$ the equation (13) also holds true and it turns into the smooth fit condition (iii) proved later on.

Furthermore, since $\mathbb{P}_{t,x}[\tau^* < T - t] > 0$ (see Lemma 1), the expression (13) shows that $\partial_x V < 0$ and therefore $x \mapsto V(t, x)$ is strictly decreasing for all $t \in [0, T]$.

(iii) Take a pair $(t, x) \in [0, T] \times \mathbb{R}$ lying on the OSB, i.e., $x = b(t)$, and consider $\varepsilon > 0$. Since $(t, x) \in D$ and $(t, x + \varepsilon) \in C$, we have that $V(t, x) = G(x)$ and $V(t, x + \varepsilon) > G(x + \varepsilon)$. Thus, taking into account the inequality (28), we get $\varepsilon^{-1}(V(t, x + \varepsilon) - V(t, x)) > \varepsilon^{-1}(G(x + \varepsilon) - G(x)) \geq -1$, which, after making $\varepsilon \rightarrow 0$ turns into $\partial_x^+ V(t, x) \geq -1$. On the other hand, by considering the optimal stopping time $\tau_\varepsilon := \tau^*(t, x + \varepsilon)$ and following the same

arguments showed at (29), we get that $\partial_x^+ V(t, x) \leq -1$. Therefore $\partial_x^+ V(t, b(t)) = -1$ for all $t \in [0, T)$. Since $V = G$ in D , it follows straightforwardly that $\partial_x^- V(t, b(t)) = -1$ and hence the smooth fit condition holds.

(iv) Notice that, due to (i), alongside to (4), (3), (13), and recalling that $x \mapsto V(t, x)$ is convex (and therefore $\partial_{x^2} V \geq 0$), we get that

$$\begin{aligned} \partial_t V(t, x) &\leq \lambda V(t, x) - \frac{T-t-x}{T-t} \partial_x V(t, x) \\ &= -\mathbb{E} \left[e^{-\lambda \tau^*} (x - S) \frac{1-t-\tau^*}{1-t} \left(\lambda + \frac{x-S}{1-t} \right) \right]. \end{aligned}$$

Therefore $\partial_t V \leq 0$ on the set $C_S := [0, T) \times [S, \infty) \subset C$.

For some small $\varepsilon > 0$, denote by $(X_s^{[t, T]})_{s \geq 0}^{T-t+\varepsilon}$ a process such that, for $s \in [0, T-t]$ it behaves as the Brownian bridge $X^{[t, T]}$, and the remaining part stays constant at the value S , i.e., $X_s^{[t, T]} = S$ for $s \in [T-t, T-t+\varepsilon]$. Let $\mu^{[t, T]}$ be its drift, and define the process $(X_s^{[t-\varepsilon, T]})_{s \geq 0}^{T-t+\varepsilon} = X^{[t-\varepsilon, T]}$ with drift $\mu^{[t-\varepsilon, T]}$. Since $\mu^{[t, T]}(t, x) \geq \mu^{[t-\varepsilon, T]}$ whenever $x \leq S$, Theorem 1.1 from Ikeda and Watanabe (1977) guarantees that $X_s^{[t, T]} \geq X_s^{[t-\varepsilon, T]}$ $\mathbb{P}_{t,x}$ -a.s., for all (t, x) and for all $s \leq \tau^S$, where $\tau^S := \inf\{s \in [0, T-t] : X_s^{[t, T]} > S\}$.

Assume now that both processes start at $x \leq S$, and consider the stopping time $\tau^* = \tau^*(t, x)$. Therefore, since $V(t+s \wedge \tau^*, X_{s \wedge \tau^*}^{[t, T]})$ and $V(t+s, X_s^{[t-\varepsilon, T]})$ are a martingale and a supermartingale (see Section 2.2 from Peskir and Shiryaev (2006)), respectively, we have

$$\begin{aligned} V(t, x) - V(t-\varepsilon, x) &\leq \mathbb{E} \left[V(t+\tau^S \wedge \tau^*, X_{\tau^S \wedge \tau^*}^{[t, T]}) - V(t-\varepsilon+\tau^S \wedge \tau^*, X_{\tau^S \wedge \tau^*}^{[t-\varepsilon, T]}) \right] \\ &\leq \mathbb{E} \left[\mathbb{I}(\tau^* \leq \tau^S, \tau^* < T-t) e^{-\lambda \tau^*} (X_{\tau^*}^{[t-\varepsilon, T]} - X_{\tau^*}^{[t, T]})^+ \right] \\ &\quad + \mathbb{E} \left[\mathbb{I}(\tau^* \wedge \tau^S = T-t) e^{-\lambda \tau^*} (X_{T-t}^{[t-\varepsilon, T]} - X_{T-t}^{[t, T]})^+ \right] \\ &\quad + \mathbb{E} \left[\mathbb{I}(\tau^S \leq \tau^*, \tau^S < T-t) V(t+\tau^S, X_{\tau^S}^{[t, T]}) - V(t-\varepsilon+\tau^S, X_{\tau^S}^{[t-\varepsilon, T]}) \right] \\ &\leq \mathbb{E} \left[\mathbb{I}(\tau^S \leq \tau^*, \tau^S < T-t) V(t-\varepsilon+\tau^S, S) - V(t-\varepsilon+\tau^S, X_{\tau^S}^{[t-\varepsilon, T]}) \right] \\ &\leq 0, \end{aligned}$$

where the second inequality comes after noticing that τ^* is not optimal for $X^{[t-\varepsilon, T]}$ and using (28); the third inequality holds since $X_{\tau^*}^{[t-\varepsilon, T]} \leq X_{\tau^*}^{[t, T]}$ for $\tau^* \leq \tau^S$, $X_{T-t}^{[t-\varepsilon, T]} \leq X_{T-t}^{[t, T]}$ whenever $\tau^* \wedge \tau^S = T-t$, and the fact that $\partial_t V \leq 0$ on the set C_S ; and the last inequality

relies on the increasing behavior of $x \mapsto V(t, x)$. Finally, after dividing by ε and taking $\varepsilon \rightarrow 0$, we get the claimed result. Note that a similar argument for a different gain function has recently appeared in Tiziano and Milazzo (2019).

(v) Let $(X_{t_i+s}^{[t_i, T]})_{s \geq 0}^{[0, T-t_i]}$ be a Brownian bridge going from $X_{t_i} = x$ to $X_T = S$ for any $x \in \mathbb{R}$, with $i = 1, 2$. Notice that, according to (3), the following holds:

$$X_{t_2+s'}^{[t_2, T]} \stackrel{d}{=} r^{1/2} X_{t_1+s}^{[t_1, T]} + (1 - r^{1/2})(S - x) \frac{s}{T - t_1}, \quad (30)$$

where $r = \frac{T-t_2}{T-t_1}$, $s \in [0, T - t_1]$, and $s' = sr \in [0, T - t_2]$.

Take $0 \leq t_1 < t_2 < T$, consider $\tau_1 := \tau^*(t_1, x)$, and set $\tau_2 := \tau_1 r$. Since $t \mapsto V(t, x)$ is decreasing for every $x \in \mathbb{R}$, then

$$\begin{aligned} 0 &\leq V(t_1, x) - V(t_2, x) \\ &\leq \mathbb{E}_{t_1, x} \left[e^{-\lambda \tau_1} G \left(X_{t_1+\tau_1}^{[t_1, T]} \right) \right] - \mathbb{E}_{t_2, x} \left[e^{-\lambda \tau_2} G \left(X_{t_2+\tau_2}^{[t_2, T]} \right) \right] \\ &\leq \mathbb{E} \left[e^{-\lambda \tau_2} \left(G \left(X_{t_1+\tau_1}^{[t_1, T]} \right) - G \left(X_{t_2+\tau_2}^{[t_2, T]} \right) \right) \right] \\ &\leq \mathbb{E} \left[\left(X_{t_2+\tau_2}^{[t_2, T]} - X_{t_1+\tau_1}^{[t_1, T]} \right)^+ \right] \\ &= \mathbb{E} \left[\left((r^{1/2} - 1) \left(X_{t_1+\tau_1}^{[t_1, T]} + (S - x) \frac{\tau_1}{T - t_1} \right) \right)^+ \right] \\ &\leq ((r^{1/2} - 1) (S + \mathbb{I}(x \leq S)(S - x)))^+, \end{aligned}$$

where the first equality comes after applying (30) and the last inequality takes place since $r < 1$ and $X_{t_1+\tau_1}^{[t_1, T]} \leq S$.

Hence, $V(t_1, x) - V(t_2, x) \rightarrow 0$ as $t_1 \rightarrow t_2$, i.e., $t \mapsto V(t, x)$ is continuous for every $x \in \mathbb{R}$, and thus, to address the continuity of V is sufficient to prove that, for a fixed t , $x \mapsto V(t, x)$ is uniformly continuous within a neighborhood of t . The latter comes after the following inequality, which comes right after applying similar arguments to those used in (27):

$$\begin{aligned} 0 \leq V(t, x_1) - V(t, x_2) &\leq (x_2 - x_1) \mathbb{E} \left[e^{-\lambda \tau^*} \frac{T - t - \tau^*}{T - t} \right] \\ &\leq x_2 - x_1, \end{aligned}$$

where $x_1, x_2 \in \mathbb{R}$ are such that $x_1 \leq x_2$ and $\tau^* = \tau^*(t, x_1)$. ■

Proof of Proposition 3. We already proved the right-continuity of b in Proposition 1, so this proof is devoted to prove its left-continuity.

Let us assume that b is not left-continuous. Therefore, as b is non-decreasing, we can ensure the existence of a point $t_* \in (0, T)$ such that $b(t_*^-) < b(t_*)$, which allows us to take x' in the interval $(b(t_*^-), b(t_*))$ and consider the right-open rectangle $\mathcal{R} = [t', t_*) \times [b(t_*^-), x'] \subset C$ (see illustration of Figure 10), with $t' \in (0, t_*)$.

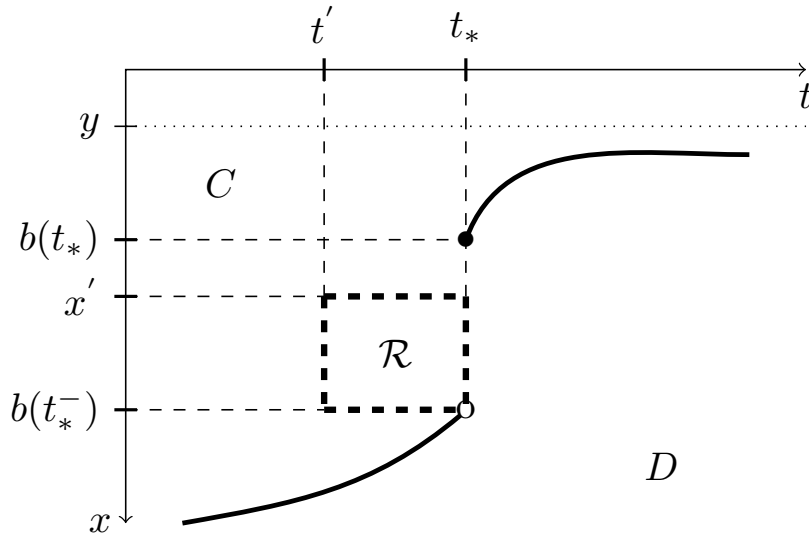


Figure 10: Graphical sketch of the proof of left continuity of b .

Applying twice the fundamental theorem of calculus, using that $(t, b(t)) \in D$ for all $t \in [0, T]$, the smooth fit condition (iii), and the fact that $x \mapsto V(t, x)$ is \mathcal{C}^2 on C , we obtain

$$V(t, x) - G(x) = \int_{b(t)}^x \int_{b(t)}^u (\partial_{x^2} V(t, v) - \partial_{x^2} G(v)) dv du, \quad (31)$$

for all $(t, x) \in \mathcal{R}$.

On the other hand, if we set $m := -\sup_{(t,x) \in \mathcal{R}} \partial_x V(t, x)$, then we readily obtain from (13) that $m > 0$ (see Lemma 1), which, combined with $\partial_t V + \mathbb{L}_X V = \lambda V$ on C and $\partial_t V \leq 0$ on C ((i) and (iv) from Proposition 2), along with the fact that $V(t, x) \geq 0$ for

all admissible pairs (t, x) , gives

$$\begin{aligned}\partial_{x^2}V(t, x) &= \frac{2}{\sigma^2} \left(\lambda V(t, x) - \frac{S-x}{T-t} \partial_x V(t, x) - \partial_t V(t, x) \right) \\ &\geq \frac{2m}{\sigma^2} \frac{S-x}{T-t} > 0,\end{aligned}\tag{32}$$

for all $(t, x) \in \mathcal{R}$. Therefore, by noticing that $\partial_{x^2}G(x) = 0$ for all $x \in (b(t_*^-), x')$ and plugging-in (32) into (31), we get

$$\begin{aligned}V(t, x) - G(x) &\geq \int_{b(t)}^x \int_{b(t)}^u \frac{2m}{\sigma^2} \frac{S-x}{T-t} dv du \\ &\geq \frac{2m}{\sigma^2} \frac{S-x}{T-t} \int_{b(t_*^-)}^x \int_{b(t_*^-)}^u dv du \\ &= \frac{2m}{\sigma^2} \frac{S-x}{T-t} (x - b(t_*^-))^2.\end{aligned}$$

Finally, after taking $t \rightarrow t_*$ on both sides of the above equation, we obtain $V(t_*, x) - G(x) > 0$ for all $x \in (b(t_*^-), b(t_*))$, which contradicts the fact that $(t_*, x) \in D$. ■

Proof of Theorem 1.

Assume we have a function $c : [0, T] \rightarrow \mathbb{R}$ that solves the integral equation (20) and define the function

$$\begin{aligned}V^c(t, x) &= \int_t^T e^{-\lambda(u-t)} \left(\frac{1}{T-u} + \lambda \right) \mathbb{E}_{t,x} [(S - X_u) \mathbb{1}(X_u \leq c(u))] du \\ &= \int_t^T K_{\sigma,\lambda}(t, x, u, c(u)) du,\end{aligned}\tag{33}$$

where $X = (X_s)_{s=0}^T$ is a Brownian bridge with σ volatility that ends at $X_T = S$, and $K_{\sigma,\lambda}$ is defined at (19). It turns out that $x \mapsto K_{\sigma,\lambda}(t, x, u, c(u))$ is twice continuously differentiable and therefore differentiating inside the integral symbol at (33) yields $\partial_x V^c(t, x)$ and $\partial_{x^2} V^c(t, x)$, and furthermore ensures their continuity on $[0, T) \times \mathbb{R}$.

Let us compute the operator $\partial_t + \mathbb{L}_X$ acting on the function V^c ,

$$\partial_t V^c + \mathbb{L}_X V^c(t, x) = \lim_{h \downarrow 0} \frac{\mathbb{E}_{t,x}[V^c(t+h, X_{t+h})] - V^c(t, x)}{h}.$$

Define the function

$$I(t, u, x_1, x_2) := e^{-\lambda(u-t)} \left(\frac{1}{T-u} + \lambda \right) (S - x_1) \mathbb{1}(x_1 \leq x_2)\tag{34}$$

and notice that

$$\begin{aligned}
\mathbb{E}_{t,x}[V^c(t+h, X_{t+h})] &= \mathbb{E}_{t,x} \left[\mathbb{E}_{t+h, X_{t+h}} \left[\int_{t+h}^T I(t+h, u, X_u, c(u)) \, du \right] \right] \\
&= \mathbb{E}_{t,x} \left[\mathbb{E}_{t,x} \left[\int_{t+h}^T I(t+h, u, X_u, c(u)) \, du \mid \mathcal{F}_{t+h} \right] \right] \\
&= \mathbb{E}_{t,x} \left[\int_{t+h}^T I(t+h, u, X_u, c(u)) \, du \right],
\end{aligned}$$

where $(\mathcal{F}_s)_{s=0}^T$ is the natural filtration of X . Therefore,

$$\begin{aligned}
&\partial_t V^c + \mathbb{L}_X V^c(t, x) \\
&= \lim_{h \downarrow 0} \frac{\mathbb{E}_{t,x} \left[\int_{t+h}^T I(t+h, u, X_u, c(u)) \, du \right] - \mathbb{E}_{t,x} \left[\int_t^T I(t, u, X_u, c(u)) \, du \right]}{h} \\
&= \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}_{t,x} \left[\int_{t+h}^T (e^{\lambda h} - 1) I(t, u, X_u, c(u)) \, du \right] - \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}_{t,x} \left[\int_t^{t+h} I(t, u, X_u, c(u)) \, du \right] \\
&= \lambda V(t, x) - (S - x) \left(\frac{1}{T-t} + \lambda \right) \mathbb{1}(x \leq c(t)).
\end{aligned}$$

From this result, alongside with (4) and the fact that V^c , $\partial_x V^c$, and $\partial_{x^2} V^c$ are continuous on $[0, T) \times \mathbb{R}$, we get the continuity of $\partial_t V^c$ on $C_1 \cup C_2$, where

$$\begin{aligned}
C_1 &:= \{(t, x) \in [0, T) \times \mathbb{R} : x > c(t)\}, \\
C_2 &:= \{(t, x) \in [0, T) \times \mathbb{R} : x < c(t)\}.
\end{aligned}$$

Now define the function $F^{(t)}(s, x) := e^{-\lambda s} V^c(t+s, x)$ with $s \in [0, T-t)$, $x \in \mathbb{R}$, and consider the sets

$$\begin{aligned}
C_1^t &:= \{(s, x) \in C_1 : t \leq s < T\}, \\
C_2^t &:= \{(s, x) \in C_2 : t \leq s < T\}.
\end{aligned}$$

We claim that $F^{(t)}$ satisfies the (iii-b) version of the hypothesis of Lemma 2 taking $C = C_1^t$ and $D^\circ = C_2^t$. Indeed: $F^{(t)}$, $\partial_x F^{(t)}$, and $\partial_{x^2} F^{(t)}$ are continuous on $[0, T) \times \mathbb{R}$; it has been proved that $F^{(t)}$ is $\mathcal{C}^{1,2}$ on C_1^t and C_2^t ; we are assuming that c is a continuous function of bounded variation; and $(\partial_t F^{(t)} + \mathbb{L}_X F^{(t)})(s, x) = -e^{-\lambda s} (S - x) \left(\frac{1}{T-t-s} + \lambda \right) \mathbb{1}(x \leq c(t+s))$ is locally bounded on $C_1^t \cup C_2^t$.

Thereby, we can use the (iii-b) version of Lemma 2 to obtain the following change of variable formula, which is missing the local time term because of the continuity of F_x on $[0, T) \times \mathbb{R}$:

$$\begin{aligned} e^{-\lambda s} V^c(t+s, X_{t+s}) \\ = V^c(t, x) - \int_t^{t+s} e^{-\lambda(u-t)} (S - X_u) \left(\frac{1}{T-u} + \lambda \right) \mathbb{1}(X_u \leq c(u)) du + M_s^{(1)}, \end{aligned} \quad (35)$$

with $M_s^{(1)} = \int_t^{t+s} e^{-\lambda(u-t)} \sigma \partial_x V^c(u, X_u) dB_u$. Notice that $(M_s^{(1)})_{s=0}^{T-t}$ is a martingale under $\mathbb{P}_{t,x}$.

In the same way, we can apply the (iii-b) version of Lemma 2 this time using the function $F(s, x) = e^{-\lambda s} G(X_{t+s})$, and taking $C = \{(s, x) \in [0, T-t) \times \mathbb{R} : x > S\}$ and $D^\circ = \{(s, x) \in [0, T-t) \times \mathbb{R} : x < S\}$, thereby getting

$$\begin{aligned} e^{-\lambda s} G(X_{t+s}) = G(x) - \int_t^{t+s} e^{-\lambda(u-t)} (S - X_u) \left(\frac{1}{T-u} + \lambda \right) \mathbb{1}(X_u < S) du \\ - M_s^{(2)} + \frac{1}{2} \int_t^{t+s} e^{-\lambda(u-t)} \mathbb{1}(X_u = S) dl_s^S(X), \end{aligned} \quad (36)$$

where $M_s^{(2)} = \sigma \int_t^{t+s} e^{-\lambda(u-t)} \mathbb{1}(X_u < S) dB_u$, with $0 \leq s \leq T-t$, is a martingale under $\mathbb{P}_{t,x}$.

Consider the following stopping time for (t, x) such that $x \leq c(t)$:

$$\rho_c := \inf \{0 \leq s \leq T-t : X_{t+s} \geq c(t+s) \mid X_t = x\}. \quad (37)$$

In this way, along with assumption $c(t) < S$ for all $t \in (0, T)$, we can ensure that $\mathbb{1}(X_{t+s} \leq c(t+s)) = \mathbb{1}(X_{t+s} \leq S) = 1$ for all $s \in [0, \rho_c)$, as well as $\int_t^{t+s} e^{-\lambda(u-t)} \mathbb{1}(X_u = S) dl_s^S(X) = 0$. Recall that $V^c(t, c(t)) = G(c(t))$ for all $t \in [c, T)$ since c solves the integral equation (20). Moreover, $V^c(T, S) = 0 = G(S)$. Hence, $V^c(t + \rho_c, X_{t+\rho_c}) = G(X_{t+\rho_c})$. Therefore, we are

able now to derive the following relation from equations (35) and (36):

$$\begin{aligned}
V^c(t, x) &= \mathbb{E}_{t,x}[e^{-\lambda\rho_c}V^c(t + \rho_c, X_{t+\rho_c})] \\
&\quad + \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}(S - X_u) \left(\frac{1}{T-u} + \lambda\right) \mathbb{1}(X_u \leq c(u)) \, du\right] \\
&= \mathbb{E}_{t,x}[e^{-\lambda\rho_c}G(X_{t+\rho_c})] \\
&\quad + \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}(S - X_u) \left(\frac{1}{T-u} + \lambda\right) \mathbb{1}(X_u \leq S) \, du\right] \\
&= G(x).
\end{aligned}$$

The vanishing of the martingales $M_{\rho_c}^{(1)}$ and $M_{\rho_c}^{(2)}$ comes after using the optional stopping theorem (see, e.g., Section 3.2 from Peskir and Shiryaev (2006)). Therefore, we have just proved that $V^c = G$ on C_2 .

Now define the stopping time

$$\tau_c := \inf\{0 \leq u \leq T - t : X_{t+u} \leq c(t+u) \mid X_t = x\}$$

and plug-in it into equation (35) to obtain the expression

$$\begin{aligned}
V^c(t, x) &= e^{-\lambda\tau_c}V^c(t + \tau_c, X_{t+\tau_c}) \\
&\quad + \int_t^{t+\tau_c} e^{-\lambda(u-t)}(S - X_u) \left(\frac{1}{T-u} + \lambda\right) \mathbb{1}(X_u \leq c(u)) \, du - M_{\tau_c}^{(1)}.
\end{aligned}$$

Notice that, due to the definition of τ_c , $\mathbb{1}(X_{t+u} \leq c(t+u)) = 0$ for all $0 \leq u < \tau_c$ whenever $\tau_c > 0$ (the case $\tau_c = 0$ is trivial). In addition, the optional sampling theorem ensures that $\mathbb{E}_{t,x}[M_{\tau_c}^{(1)}] = 0$. Therefore, the following formula comes after taking $\mathbb{P}_{t,x}$ -expectation in the above equation and considering that $V^c = G$ on C_2 :

$$V^c(t, x) = \mathbb{E}_{t,x}[e^{-\lambda\tau_c}V^c(t + \tau_c, X_{t+\tau_c})] = \mathbb{E}_{t,x}[e^{-\lambda\tau_c}G(X_{t+\tau_c})],$$

for all $(t, x) \in [0, T) \times \mathbb{R}$. Recalling the definition of V from (1), we realize that the above equality leads to

$$V^c(t, x) \leq V(t, x), \tag{38}$$

for all $(t, x) \in [0, T) \times \mathbb{R}$.

Take $(t, x) \in C_2$ satisfying $x < \min\{b(t), c(t)\}$, where b is the OSB for (1), and consider the stopping time ρ_c defined as

$$\rho_b := \inf \{0 \leq s \leq T - t : X_{t+s} \geq b(t+s) \mid X_t = x\}.$$

Since $V = G$ on D , the following equality holds true due to (14) and from noticing that $\mathbb{1}(X_{t+u} \leq b(t+u)) = 1$ for all $0 \leq u < \rho_b$:

$$\mathbb{E}_{t,x}[e^{-\lambda\rho_b}V(t+\rho_b, X_{t+\rho_b})] = G(x) - \mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}(S - X_u) \left(\frac{1}{T-u} + \lambda\right) du\right].$$

On the other hand, we get the next equation after substituting s for ρ_b at (35) and recalling that $V = G$ on C_2 :

$$\begin{aligned} & \mathbb{E}_{t,x}[e^{-\lambda\rho_b}V(t+\rho_b, X_{t+\rho_b})] \\ &= G(x) - \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}(S - X_u) \left(\frac{1}{T-u} + \lambda\right) \mathbb{1}(X_u \leq c(u)) du\right]. \end{aligned}$$

Therefore, we can use (38) to merge the two previous equalities into

$$\begin{aligned} & \mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}(S - X_u) \left(\frac{1}{T-u} - \lambda\right) \mathbb{1}(X_u \leq c(u)) du\right] \\ & \geq \mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}(S - X_u) \left(\frac{1}{T-u} - \lambda\right) du\right], \end{aligned}$$

meaning that $b(t) \leq c(t)$ for all $t \in [0, T]$ since c is continuous.

Suppose there exists a point $t \in (0, T)$ such that $b(t) < c(t)$ and fix $x \in (b(t), c(t))$. Consider the stopping time

$$\tau_b := \inf\{0 \leq u \leq T - t : X_{t+u} \leq b(t+u) \mid X_t = x\}$$

and plug-in it both into (14) and (35) replacing s before taking the $\mathbb{P}_{t,x}$ -expectation. We obtain

$$\begin{aligned} & \mathbb{E}_{t,x}[e^{-\lambda\tau_b}V^c(t+\tau_b, X_{t+\tau_b})] \\ &= \mathbb{E}_{t,x}[e^{-\lambda\tau_b}G(X_{t+\tau_b})] \\ &= V^c(t, x) - \mathbb{E}_{t,x}\left[\int_t^{t+\tau_b} e^{-\lambda(u-t)}(S - X_u) \left(\frac{1}{T-u} + \lambda\right) \mathbb{1}(X_u \leq c(u)) du\right] \end{aligned}$$

and

$$\mathbb{E}_{t,x}[e^{-\lambda\tau_b}V(t+\tau_b, X_{t+\tau_b})] = \mathbb{E}_{t,x}[e^{-\lambda\tau_b}G(X_{t+\tau_b})] = V(t, x).$$

Thus, from (38) we get

$$\mathbb{E}_{t,x} \left[\int_t^{t+\tau_b} e^{-\lambda(u-t)} (S - X_u) \left(\frac{1}{T-u} + \lambda \right) \mathbb{1}(X_u \leq c(u)) du \right] \leq 0.$$

Using the fact that $x > b(t)$ and the time-continuity of the process X , we can state that $\tau_b > 0$. Therefore, the previous inequality can only happen if $\mathbb{1}(X_s \leq c(s)) = 0$ for all $t \leq s \leq t + \tau_b$, meaning that $b(s) \geq c(s)$ for all $t \leq s \leq t + \tau_b$, which contradicts the assumption $b(t) < c(t)$. ■

Proof of Proposition 4. Since the OSP (21) satisfies the hypothesis stated in Corollary 2.9 from Peskir and Shiryaev (2006) (V_i lower semi-continuous and G_i upper semi-continuous), we can ensure the existence of the OSP $\tau_i^*(t, x)$ defined at (6) for the pair (t, x) , where $i = 1, 2$. Moreover, Theorem 2.4 from Peskir and Shiryaev (2006) guarantees that $\mathbb{P}_{t,x}^{(i)}[\tau_i^*(t, x) \leq \tau_*] = 1$ for any other OST τ_* of the OSP (21), where $\mathbb{P}_{t,x}^{(i)}$ denotes the law such that $\mathbb{P}_{t,x}^{(i)}[X_t^{(i)} = x] = 1$.

(i) Define the sets $D_i^{\alpha,A} := \{(t, x) \in [0, T] \times \mathbb{R} : (t, \alpha^{-1}(x - A)) \in D_i\}$ for $i = 1, 2$, and notice that $\tau_1^*(t, x) \stackrel{d}{=} \inf\{0 \leq s \leq T - t : X_{t+s}^{(2)} \in D_1^{\alpha,A} \mid X_t^{(2)} = \alpha x + A\}$ as well as $\tau_2^*(t, \alpha x + A) \stackrel{d}{=} \inf\{0 \leq s \leq T - t : X_{t+s}^{(1)} \in D_2^{\alpha^{-1}, -\alpha^{-1}A} \mid X_t^{(1)} = x\}$, for $(t, x) \in [0, T] \times \mathbb{R}$. Suppose that

$$\mathbb{E}_{t, \alpha x + A} \left[e^{-\lambda\tau_2^*(t, \alpha x + A)} G_2 \left(X_{t+\tau_2^*(t, \alpha x + A)}^{(2)} \right) \right] > \mathbb{E}_{t, \alpha x + A} \left[e^{-\lambda\tau_1^*(t, x)} G_2 \left(X_{t+\tau_1^*(t, x)}^{(2)} \right) \right].$$

Then,

$$\begin{aligned} \mathbb{E}_{t,x} \left[e^{-\lambda\tau_2^*(t, \alpha x + A)} G_1 \left(X_{t+\tau_2^*(t, \alpha x + A)}^{(1)} \right) \right] &= \mathbb{E}_{t, \alpha x + A} \left[e^{-\lambda\tau_2^*(t, \alpha x + A)} G_2 \left(X_{t+\tau_2^*(t, \alpha x + A)}^{(2)} \right) \right] \\ &> \mathbb{E}_{t, \alpha x + A} \left[e^{-\lambda\tau_1^*(t, x)} G_2 \left(X_{t+\tau_1^*(t, x)}^{(2)} \right) \right] \\ &= \mathbb{E}_{t,x} \left[e^{-\lambda\tau_1^*(t, x)} G_1 \left(X_{t+\tau_1^*(t, x)}^{(1)} \right) \right] \end{aligned}$$

which is a contradiction. Therefore, our original assumption has to be wrong, meaning that $\tau_2^*(t, \alpha x + A) \leq \tau_1^*(t, x)$ $\mathbb{P}_{t,x}^{(1)}$ -a.s. as well as $\mathbb{P}_{t, \alpha x + A}^{(2)}$ -a.s. (notice that $\mathbb{P}_{t,x}^{(1)} = \mathbb{P}_{t, \alpha x + A}^{(2)}$).

Interchanging the roles of $t_1^*(t, x)$ and $t_2^*(t, \alpha x + A)$ along the argumentation given above, and making the corresponding rearrangements, we get the opposite inequality. Thus, since both D_1 and D_2 are closed sets, then $D_2 = D_1^{\alpha, A}$ or, reciprocally, $D_1 = D_2^{\alpha^{-1}, -\alpha^{-1}A}$.

(ii) Fix $(t, x) \in [0, T] \times \mathbb{R}$ and let $\tau_1^* = \tau_1^*(t, x)$ as well as $\tau_2^* = \tau_2^*(t, x)$. Notice that $\tau_1^* \stackrel{d}{=} \inf\{0 \leq s \leq T - t : X_{t+s}^{(2)} \in D_1 \mid X_t^{(2)} = x\}$ and $\tau_2^* \stackrel{d}{=} \inf\{0 \leq s \leq T - t : X_{t+s}^{(1)} \in D_2 \mid X_t^{(1)} = x\}$. Suppose that

$$\mathbb{E}_{t,x} \left[e^{-\lambda \tau_2^*} G_2 \left(X_{t+\tau_2^*}^{(2)} \right) \right] > \mathbb{E}_{t,x} \left[e^{-\lambda \tau_1^*} G_2 \left(X_{t+\tau_1^*}^{(2)} \right) \right].$$

Since $G_1 = G_2$ on $D_1 \cup D_2$, then

$$\begin{aligned} \mathbb{E}_{t,x} \left[e^{-\lambda \tau_2^*} G_1 \left(X_{t+\tau_2^*}^{(1)} \right) \right] &= \mathbb{E}_{t,x} \left[e^{-\lambda \tau_2^*} G_2 \left(X_{t+\tau_2^*}^{(2)} \right) \right] \\ &> \mathbb{E}_{t,x} \left[e^{-\lambda \tau_1^*} G_2 \left(X_{t+\tau_1^*}^{(2)} \right) \right] \\ &= \mathbb{E}_{t,x} \left[e^{-\lambda \tau_1^*} G_1 \left(X_{t+\tau_1^*}^{(1)} \right) \right], \end{aligned}$$

which is an absurd and hence our assumption is wrong, this is, $\tau_2^* \leq \tau_1^*$ $\mathbb{P}_{t,x}^{(1)}$ -a.s. as well as $\mathbb{P}_{t,x}^{(2)}$ -a.s. (notice that $\mathbb{P}_{t,x}^{(1)} = \mathbb{P}_{t,x}^{(2)}$).

Swapping the roles of t_1^* and t_2^* throughout the argumentation given above, and making the correspondent rearrangements, we get the opposite inequality. Therefore, since both D_1 and D_2 are closed sets, then $D_2 = D_1$. ■

Proof of Corollary 1. First, notice that in both scenarios, (i) and (ii), the conditions G_i being upper semi-continuous and V_i lower semi-continuous from Proposition 4 are fulfilled due to the continuity of G_i (see Remark 2.10 from Peskir and Shiryaev (2006)).

(i) Since $G_1(2S - x) = G_2(x)$ and $\left[2S - X_{t+s}^{(1)} \mid X_t^{(1)} = x \right] \stackrel{d}{=} \left[X_{t+s}^{(2)} \mid X_t^{(2)} = 2S - x \right]$ for all $s \in [0, T - t]$, then we can apply (i) from Proposition 4 to show that $D_1 = \{(t, x) : (t, 2S - x) \in D_2\}$, and therefore $b_1 = 2S - b_2$.

(ii) Introduce the function $G(x) = S_2 - x$ and the Brownian bridge $(X_{t+s})_{s=0}^{T-t}$ such that $X_T = S_2$. Since $G(S_2 - x) = G_1(x)$ and $[X_{t+s} \mid X_t = S_2 - x] \stackrel{d}{=} [X_{t+s}^{(1)} \mid X_t^{(1)} = x]$

for $(t, x) \in [0, T] \times \mathbb{R}$, we get that $D_1 = \{(t, x) \in [0, T] \times \mathbb{R} : S_2 - x \in D\}$, and hence $b(t) = S_2 - b_1$, where D and b are, respectively, the stopping set and the OSB of the non-discounted OSP with gain function G and process $(X_{t+s})_{s=0}^{T-t}$.

Let us fix $t \in [0, T]$ and take x' satisfying $x' > S_2$. Consider $\varepsilon > 0$ such that $\varepsilon < x' - S_2$, as well as the stopping time $\tau_\varepsilon := \inf\{0 \leq s \leq T - t : X_{t+s} \leq S_2 + \varepsilon \mid X_t = x'\}$. Since our underlying Brownian bridge process $X^{(1)}$ is continuous and it takes the value S_2 at the expiration date T , then $\mathbb{P}_{t,x'}^{(1)}[\tau_\varepsilon < T - t] = 1$ and thus $V(t, x') \geq \mathbb{E}_{t,x'}[G(X_{t+\tau_\varepsilon})] = -\varepsilon > S_2 - x' = G(x')$, i.e., $(t, x') \notin D$. Therefore, $D \subset D_{S_2} := \{(t, x) \in [0, T] \times \mathbb{R} : x \leq S_2\}$.

On the other hand, recall from Proposition 1 that $D_2 \subset D_{S_2}$. Therefore, since $G(x) = G_2(x)$ for all x such that $(t, x) \in D_{S_2}$ for some $t \in [0, T]$, and $[X_{t+s} \mid X_t = x] \stackrel{d}{=} [X_{t+s}^{(2)} \mid X_t^{(2)} = x]$ for all $s \in [0, T - t]$, then we can use (ii) from Proposition 4 in order to get the relation $b_2 = b = S_2 - b_1$. ■

B Auxiliary lemmas

Lemma 1 *Let $(X_{t+s})_{s=0}^{T-t}$ be a Brownian bridge from X_t to $X_T = S$ with volatility σ , where $t \in [0, T]$. Let b be the optimal stopping boundary associated to the OSP*

$$V(t, x) = \sup_{0 \leq \tau \leq T-t} \mathbb{E}_{t,x} [e^{-\lambda \tau} G(X_{t+\tau})],$$

with $G(x) = (G - x)^+$, and $\lambda \geq 0$. Then, $\sup_{(t,x) \in \mathcal{R}} \partial_x V(t, x) < 0$, where \mathcal{R} is the set defined in the proof of Proposition 3.

Proof. Take $0 < \varepsilon < 1$, let $\tau^* = \tau^*(t, x)$, and define

$$p(t, x) := \mathbb{P}[\tau^* \leq (T - t)(1 - \varepsilon)].$$

Notice that

$$\begin{aligned}
p(t, x) &= \mathbb{P}_{t,x} \left[\min_{0 \leq s \leq (T-t)(1-\varepsilon)} \{X_{t+s} - b(s)\} < 0 \right] \\
&\geq \mathbb{P}_{t,x} \left[\min_{0 \leq s \leq (T-t)(1-\varepsilon)} X_{t+s} < b(t) \right] \\
&= \mathbb{P} \left[\min_{0 \leq s \leq (T-t)(1-\varepsilon)} \left\{ (S-x) \frac{s}{T-t} + \sigma \sqrt{\frac{T-t-s}{T-t}} W_s \right\} < b(t) - x \right] \\
&\geq \mathbb{P} \left[\min_{0 \leq s \leq (T-t)(1-\varepsilon)} \left\{ \sqrt{\frac{T-t-s}{T-t}} W_s \right\} < \sigma^{-1}(b(t) - \max\{x, S\}) \right] \\
&= \mathbb{P} \left[\min_{0 \leq s \leq (T-t)(1-\varepsilon)} \{W_s\} < \varepsilon^{-1/2} \sigma^{-1}(b(t) - \max\{x, S\}) \right] \\
&= 2\mathbb{P} \left[W_{(T-t)(1-\varepsilon)} < \varepsilon^{-1/2} \sigma^{-1}(b(t) - \max\{x, S\}) \right],
\end{aligned}$$

where the first inequality is justified since b is non-decreasing (see Proposition 1), while the last equality comes after applying the *reflection principle*. Therefore,

$$M := \inf_{(t,x) \in \mathcal{R}} p(t, x) > 0.$$

Finally, by using (13) we obtain the following relation for all $(t, x) \in \mathcal{R}$:

$$\begin{aligned}
\partial_x V(t, x) &\leq -e^{-\lambda(T-t)} \mathbb{E} \left[\frac{T-t-\tau^*}{T-t} \mathbb{1}(\tau^* \leq (T-t)(1-\varepsilon)) \right] \\
&\leq -e^{-\lambda(T-t)} \varepsilon p(t, x) \\
&\leq -e^{-\lambda(T-t)} \varepsilon M < 0.
\end{aligned}$$

■

For the sake of completeness, we formulate the following change-of-variable result by taking Theorem 3.1 from Peskir (2005a) and changing some of its hypothesis according to Remark 3.2 from Peskir (2005a). Specifically, the *(iii-a)* version of Lemma 2 comes after changing, in Peskir (2005a), (3.27) and (3.28) for the joint action of (3.26), (3.35), and (3.36). The *(iii-b)* version relaxes condition (3.35) into (3.37) in *ibid*.

Lemma 2 *Let $X = (X_t)_{t=0}^T$ be a diffusion process solving the SDE*

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dB_t, \quad 0 \leq t \leq T,$$

in the Itô's sense. Let $b : [0, T] \rightarrow \mathbb{R}$ be a continuous function of bounded variation, and let $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying

$$F \text{ is } \mathcal{C}^{1,2} \text{ on } C,$$

$$F \text{ is } \mathcal{C}^{1,2} \text{ on } D^\circ,$$

where $C = \{(t, x) \in [0, T] \times \mathbb{R} : x > b(t)\}$ and $D^\circ = \{(t, x) \in [0, T] \times \mathbb{R} : x < b(t)\}$.

Assume there exists $t \in [0, T]$ such that the following conditions are satisfied:

- (i) $\partial_t F + \mu \partial_x F + (\sigma^2/2) \partial_{x^2} F$ is locally bounded on $C \cup D^\circ$;
- (ii) the functions $s \mapsto \partial_x F(s, b(s)^\pm) := \partial_x F(s, \lim_{h \rightarrow 0+} b(s) \pm h)$ are continuous on $[0, t]$;
- (iii) and either
 - (iii-a) $x \mapsto F(s, x)$ is convex on $[b(s) - \delta, b(s)]$ and convex on $[b(s), b(s) + \delta]$ for each $s \in [0, t]$, with some $\delta > 0$, or,
 - (iii-b) $\partial_{x^2} F = G_1 + G_2$ on $C \cup D^\circ$, where G_1 is non-negative (or non-positive) and G_2 is continuous on \bar{C} and \bar{D}° .

Then, the following change-of-variable formula holds

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t (\partial_t F + \mu \partial_x F + (\sigma^2/2) \partial_{x^2} F)(s, X_s) \mathbb{1}(X_s \neq b(s)) \, ds \\ &\quad + \int_0^t (\sigma \partial_x F)(s, X_s) \mathbb{1}(X_s \neq b(s)) \, dB_s \\ &\quad + \frac{1}{2} \int_0^t (\partial_x F(s, X_s^+) - \partial_x F(s, X_s^-)) \mathbb{1}(X_s = b(s)) \, dl_s^b(X), \end{aligned}$$

where $dl_s^b(X)$ is the local time of X at the curve b up to time t , i.e.,

$$l_s^b(X) = \lim_{\varepsilon \rightarrow 0} \int_0^t \mathbb{1}(b(s) - \varepsilon \leq X_s \leq b(s) + \varepsilon) \, d\langle X, X \rangle_s, \quad (39)$$

where $\langle X, X \rangle$ is the predictable quadratic variation of X , and the limit above is meant in probability.

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